Graphs

CSCI 4360/6360 Data Science II

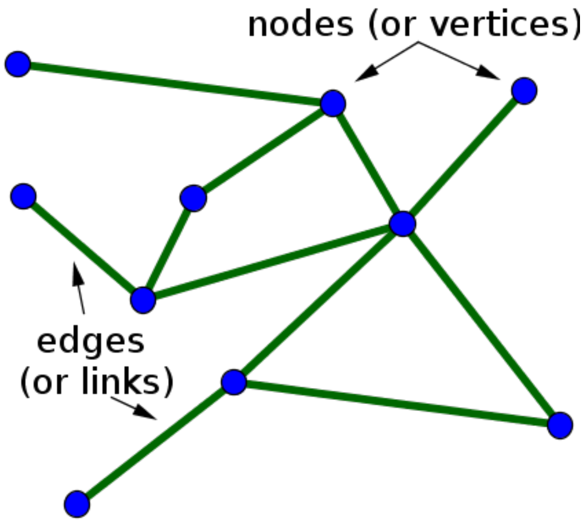
Why graphs?

- Lots of data *is* graphs
  - Facebook, Twitter, citation data, and other social networks
  - The web, the blogosphere, the semantic web, Freebase, Wikipedia, Twitter, and other *information* networks
  - Text corpora (like RCV1), large datasets with discrete feature values, and other *bipartite* networks
    - nodes = documents or words
    - links connect document → word or word → document
  - Computer networks, biological networks (proteins, ecosystems, brains, …), …
  - Heterogeneous networks with multiple types of nodes
    - people, groups, documents

Properties of Graphs

- Nodes & Edges

- Set *V* of vertices/nodes *v1, …*
- Set *E* of edges *(u,v)*,…
  - Can be weighted/directed/labeled
- *Degree of v* is # of edges on *v*
  - *Indegree* and *outdegree* for weighted graphs
- *Path* is a sequence of edges (*u1,v1),(u2,v2),…*
- *Geodesic path between u and v* is shortest path connecting them
  - Diameter is max $_{u,v \text{ in } V}$ {length of geodesic between u,v}
  - Effective diameter is 90th percentile
  - Mean diameter is over connected pairs
- *(Connected) component* is subset of nodes that are all pairwise connected via paths
- *Clique* is subset of nodes that are all pairwise connected via *edges*
- *Triangle* is a clique of size three



Properties of Graphs

- Descriptive statistics

- Number of connected components
- Diameter
- Degree distribution
- Centrality
- …

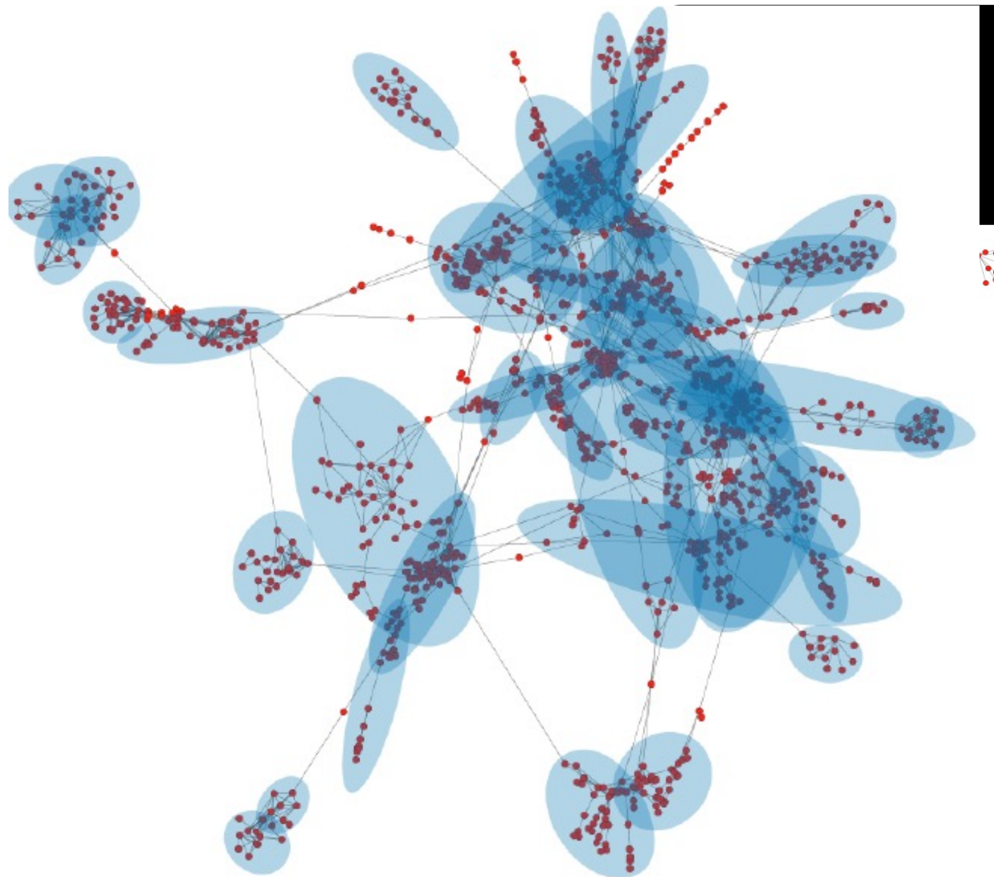
Properties of Graphs

- Models of formation and growth

- Erdos-Rayni
- Watts-Strogatz
- Preferential attachment
- Stochastic block models
- …

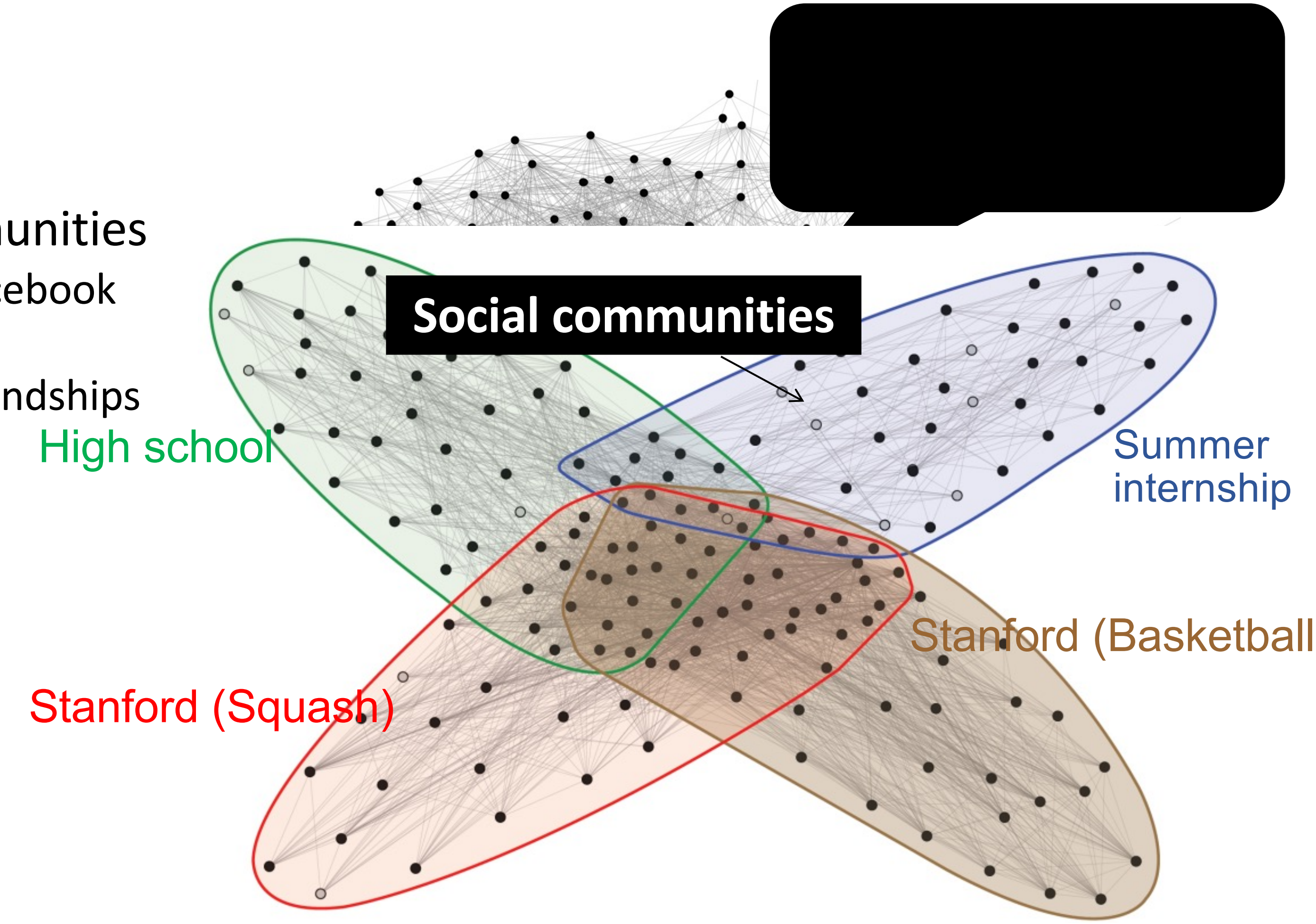
Biology

- Protein-protein interaction networks
  - Nodes: proteins
  - Edges: interactions
- Functional modules

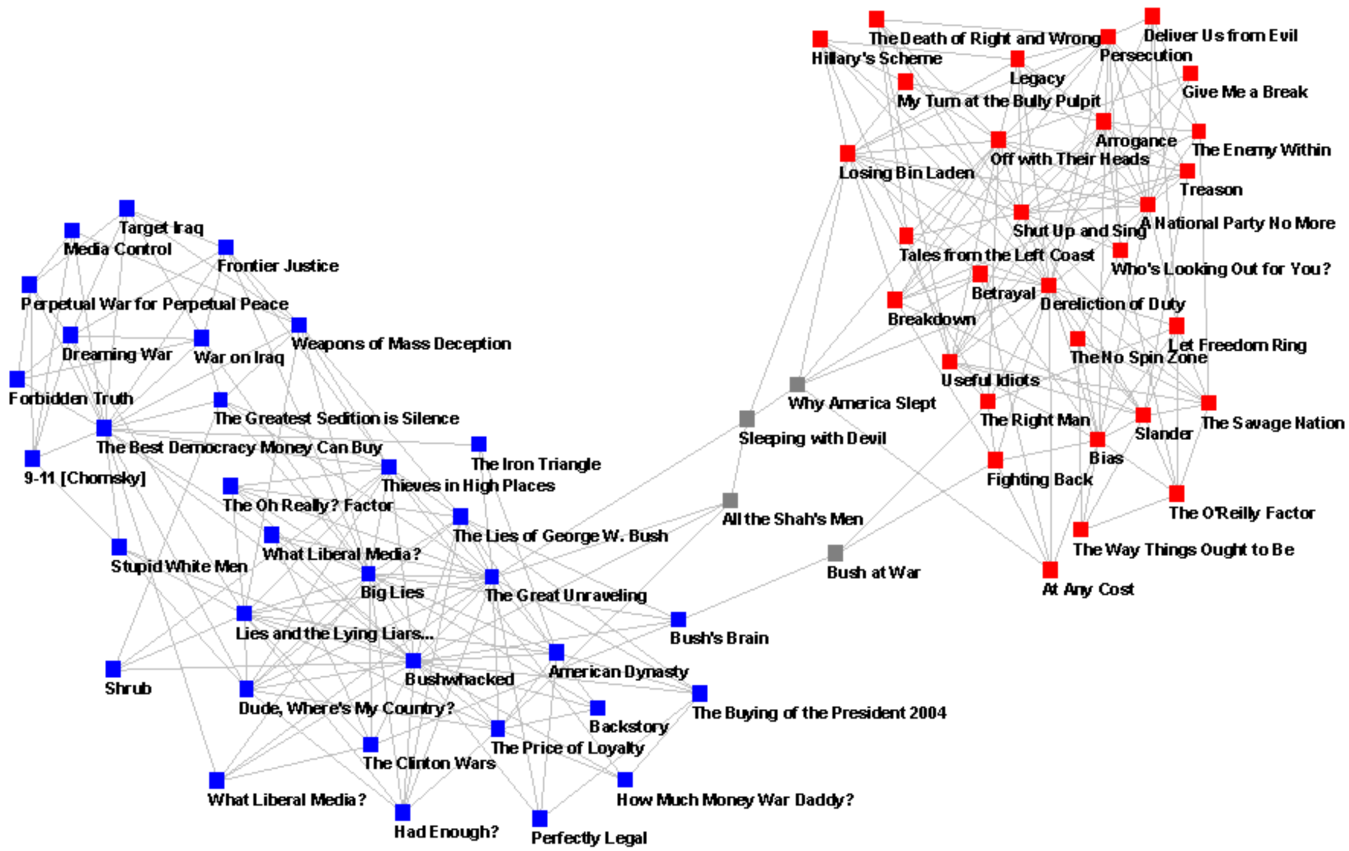


Facebook

- Social communities
  - Nodes: Facebook users
  - Edges: Friendships

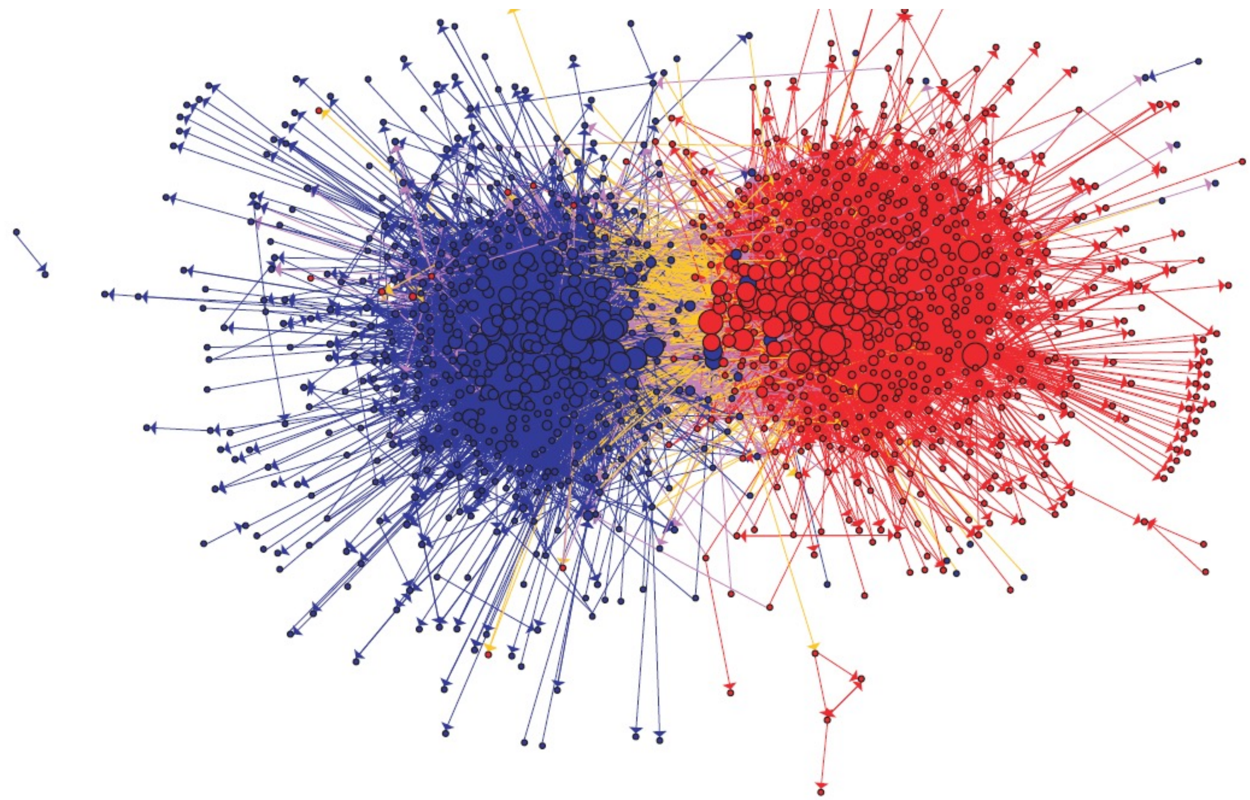


Blogs



The Death of Right and Wrong
Deliver Us from Evil
Hillary's Scheme
Persecution
Legacy
Give Me a Break
My Turn at the Bully Pulpit
Arrogance
The Enemy Within
Off with Their Heads
Losing Bin Laden
Treason
Shut Up and Sing
A National Party No More
Tales from the Left Coast
Who's Looking Out for You?
Betrayal
Dereliction of Duty
Breakdown
Let Freedom Ring
The No Spin Zone
Useful Idiots
Why America Slept
The Right Man
Slander
The Savage Nation
Sleeping with Devil
Bias
Fighting Back
The O'Reilly Factor
All the Shah's Men
The Way Things Ought to Be
Bush at War
At Any Cost
Target Iraq
Media Control
Frontier Justice
Perpetual War for Perpetual Peace
Weapons of Mass Deception
Dreaming War
War on Iraq
Forbidden Truth
The Greatest Sedition is Silence
The Best Democracy Money Can Buy
The Iron Triangle
9-11 [Chomsky]
Thieves in High Places
The Oh Really? Factor
The Lies of George W. Bush
What Liberal Media?
Stupid White Men
Big Lies
The Great Unraveling
Lies and the Lying Liars...
Bush's Brain
American Dynasty
Bushwhacked
Shrub
The Buying of the President 2004
Dude, Where's My Country?
Backstory
The Price of Loyalty
The Clinton Wars
How Much Money War Daddy?
What Liberal Media?
Had Enough?
Perfectly Legal

Blogs



Graph Models

- Fundamental graph types

Erdos-Renyi graphs

- Take *n* nodes, and connect each pair with probability *p*
  - Mean degree is *z=p(n-1)*

$$\Pr[\text{degree}(v) = k] = p_k = \boxed{\binom{n}{k} p^k} \boxed{(1-p)^{n-k}} \approx \frac{z^k e^{-z}}{k!} \quad \text{for fixed } z\text{, large } n$$

$$\boxed{e^{-z}}$$

$$\boxed{\binom{n}{k} p^k \approx \frac{n^k p^k}{k!} = \frac{(np)^k}{k!} \approx \frac{z^k}{k!}}$$

Erdos-Renyi graphs

- Take n nodes, and connect each pair with probability p
  - Mean degree is $z=p(n-1)$
  - Mean number of neighbors distance d from v is z^d
  - How large does d need to be so that $z^d >= n$?
    - If $z>1$, $d = log(n)/log(z)$
    - If $z<1$, you can't do it
  - So:
    - *There tend to be either many small components ($z<1$) or one large one ($z>1$) giant connected component)*
  - Another intuition:
    - If there are a two large connected components, then with high probability a few random edges will link them up.

Erdos-Renyi graphs

- Take n nodes, and connect each pair with probability p
  - Mean degree is $z=p(n-1)$
  - Mean number of neighbors distance d from v is z^d
  - How large does d need to be so that $z^d >= n$?
    - If $z>1$, $d = log(n)/log(z)$
    - If z<1, you can't do it
  - So:
    - *If z>1, diameters tend to be small (relative to n)*

Sociometry, Vol. 32, No. 4. (Dec., 1969), pp. 425-443.

64 of 296 chains succeed, avg chain length is 6.2

An Experimental Study of the Small World Problem*

JEFFREY TRAVERS
Harvard University

AND

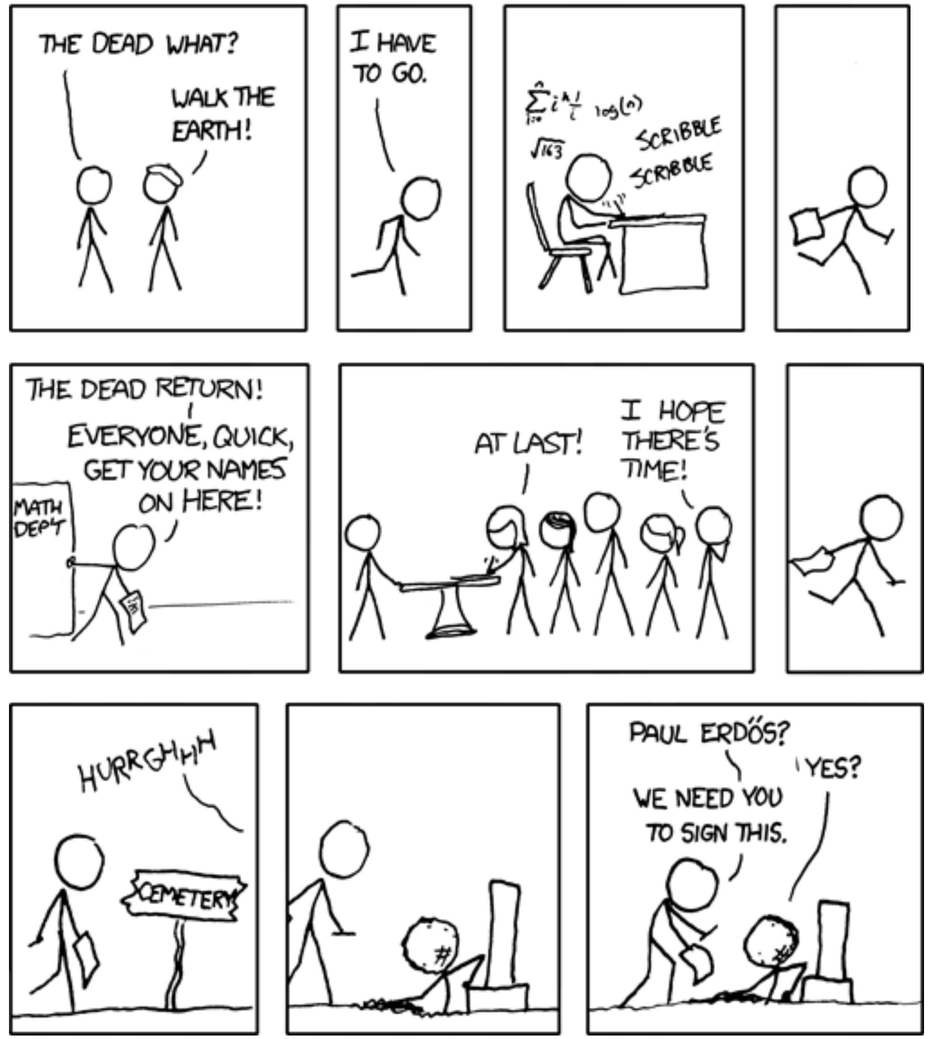
STANLEY MILGRAM
The City University of New York

Arbitrarily selected individuals (N=296) in Nebraska and Boston are asked to generate acquaintance chains to a target person in Massachusetts, employing "the small world method" (Milgram, 1967). Sixty-four chains reach the target person. Within this group the mean number of intermediaries between starters and targets is 5.2. Boston starting chains reach the target

Illustrations of the Small World

- Milgram's experiment
- Erdős numbers
  - http://www.ams.org/mathscinet/searchauthors.html
- Bacon numbers
  - http://oracleofbacon.org/
- LinkedIn
  - http://www.linkedin.com/
  - Privacy issues: the whole network is *not* visible to all


THE APOCALYPSE! THE SKIES BURN, THE SEAS TURN TO BLOOD, AND THE DEAD WALK THE EARTH!


THE DEAD WHAT?
WALK THE EARTH!
I HAVE TO GO.
SCRIBBLE
SCRIBBLE
THE DEAD RETURN!
EVERYONE, QUICK, GET YOUR NAMES ON HERE!
MATH DEPT
AT LAST!
I HOPE THERE'S TIME!
HURRGHHH
CEMETERY
PAUL ERDŐS?
YES?
WE NEED YOU TO SIGN THIS.

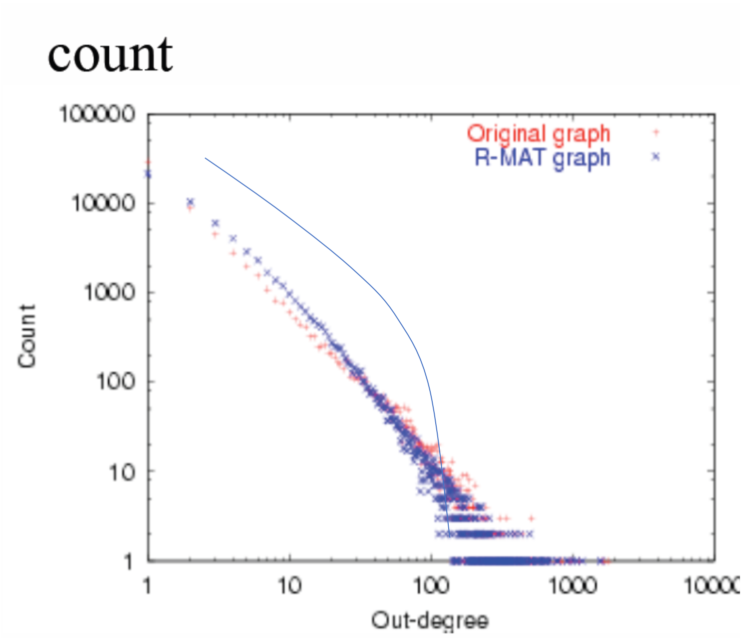
	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$
social	film actors	undirected	449 913	25 516 482	113.43	3.[illegible]8	2.3	0.20	0.78
	company directors	undirected	7 673	55 392	14.44	4.[illegible]0	–	0.59	0.88
	math coauthorship	undirected	253 339	496 489	3.92	7.[illegible]7	–	0.15	0.34
	physics coauthorship	undirected	52 909	245 300	9.27	6.[illegible]9	–	0.45	0.56
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.[illegible]2	–	0.088	0.60
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1		
	email messages	directed	59 912	86 300	1.44	4.[illegible]5	1.5/2.0		0.16
	email address books	directed	16 881	57 029	3.38	5.[illegible]2	–	0.17	0.13
	student relationships	undirected	573	477	1.66	16.[illegible]1	–	0.005	0.001
	sexual contacts	undirected	2 810				3.2		
information	WWW `nd.edu`	directed	269 504	1 497 136	5.55	11.[illegible]7	2.1/2.4	0.11	0.29
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.[illegible]8	2.1/2.7		
	citation network	directed	783 339	6 716 198	8.57		3.0/–		
	Roget's Thesaurus	directed	1 022	5 108	4.99	4.[illegible]7	–	0.13	0.15
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44
technological	Internet	undirected	10 697	31 992	5.98	3.[illegible]1	2.5	0.035	0.39
	power grid	undirected	4 941	6 594	2.67	18.[illegible]9	–	0.10	0.080
	train routes	undirected	587	19 608	66.79	2.[illegible]6	–		0.69
	software packages	directed	1 439	1 728	1.20	2.[illegible]2	1.6/1.4	0.070	0.082
	software classes	directed	1 377	2 218	1.61	1.[illegible]1	–	0.033	0.012
	electronic circuits	undirected	24 097	53 248	4.34	11.[illegible]5	3.0	0.010	0.030
	peer-to-peer network	undirected	880	1 296	1.47	4.[illegible]8	2.1	0.012	0.011
biological	metabolic network	undirected	765	3 686	9.64	2.[illegible]6	2.2	0.090	0.67
	protein interactions	undirected	2 115	2 240	2.12	6.[illegible]0	2.4	0.072	0.071
	marine food web	directed	135	598	4.43	2.[illegible]5	–	0.16	0.23
	freshwater food web	directed	92	997	10.84	1.[illegible]0	–	0.20	0.087
	neural network	directed	307	2 359	7.68	3.[illegible]7	–	0.18	0.28

Erdos-Renyi graphs

- A good model of degree distribution in ”natural” networks?

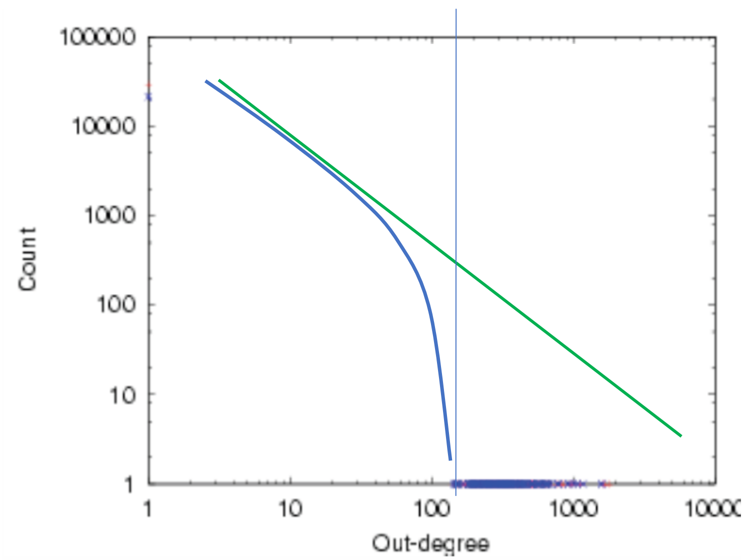
Degree distribution

- Plot cumulative degree
  - X axis is degree
  - Y axis is #nodes that have degree at least *k*
- Typically use a log-log scale
  - Straight lines are a power law; normal curve dives to zero at some point



Degree distribution

- Plot cumulative degree
  - X axis is degree
  - Y axis is #nodes that have degree at least *k*
- Typically use a log-log scale
  - Straight lines are a power law; normal curve dives to zero at some point
    - **This defines a “scale” for the network**



$$\boxed{p_k \propto k^{-\alpha}}$$

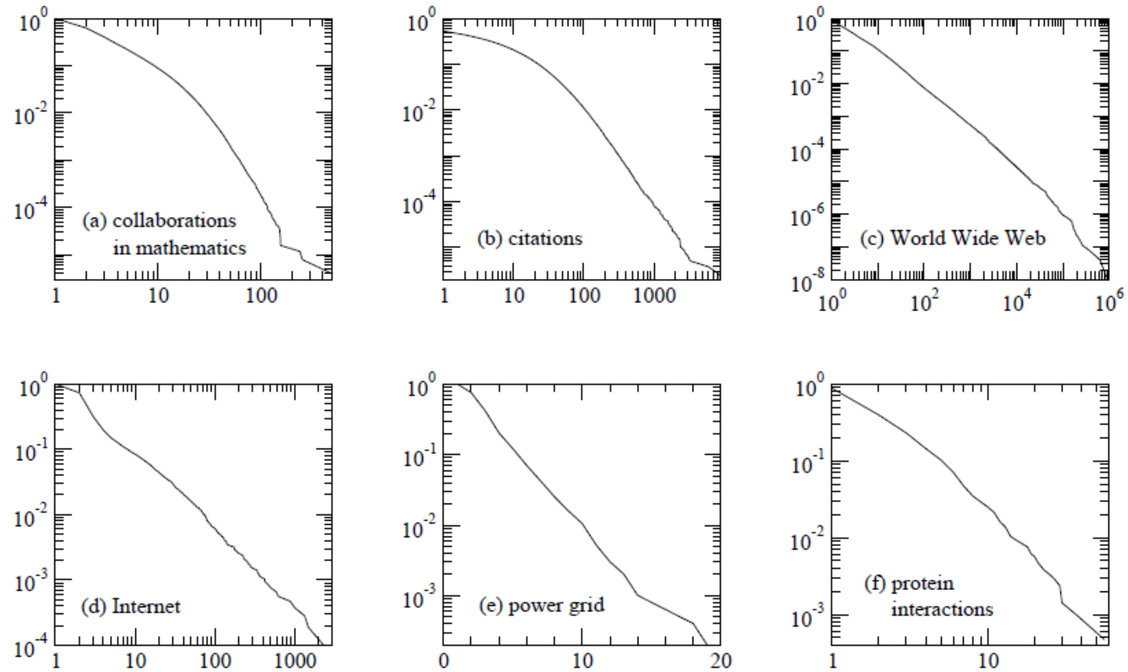


FIG. 6 Cumulative degree distributions for six different networks. The horizontal axis for each panel is vertex degree k (or in-degree for the citation and Web networks, which are directed) and the vertical axis is the cumulative probability distribution of degrees, i.e., the fraction of vertices that have degree greater than or equal to k. The networks shown are: (a) the collaboration network of mathematicians [182]; (b) citations between 1981 and 1997 to all papers cataloged by the Institute for Scientific Information [351]; (c) a 300 million vertex subset of the World Wide Web, *circa* 1999 [74]; (d) the Internet at the level of autonomous systems, April 1999 [86]; (e) the power grid of the western United States [416]; (f) the interaction network of proteins in the metabolism of the yeast *S. Cerevisiae* [212]. Of these networks, three of them, (c), (d) and (f), appear to have power-law degree distributions, as indicated by their approximately straight-line forms on the doubly logarithmic scales, and one (b) has a power-law tail but deviates markedly from power-law behavior for small degree. Network (e) has an exponential degree distribution (note the log-linear scales used in this panel) and network (a) appears to have a truncated power-law degree distribution of some type, or possibly two separate power-law regimes with different exponents.

	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1		
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001
	sexual contacts	undirected	2 810				3.2		
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7		
	citation network	directed	783 339	6 716 198	8.57		3.0/–		
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080
	train routes	undirected	587	19 603	66.79	2.16	–		0.69
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28

Graphs

- Some common properties of graphs:
  - **Distribution of node degrees**
  - Distribution of cliques (e.g., triangles)
  - **Distribution of paths**
    - **Diameter** (max shortest-path)
    - Effective diameter (90th percentile)
    - **Connected components**
  - …
- Some types of graphs to consider:
  - Real graphs (social & otherwise)
  - Generated graphs:
    - **Erdos-Renyi**
      "Bernoulli" or "Poisson"
    - Watts-Strogatz "small world" graphs
    - Barbosi-Albert "preferential attachment"
    - …

Graphs

- Some common properties of graphs:
  - **Distribution of node degrees: often scale-free**
  - Distribution of cliques (e.g., triangles)
  - **Distribution of paths**
    - **Diameter** (max shortest-path)
    - Effective **diameter** (90th percentile) **often small**
    - **Connected components usually one giant CC**
  - …

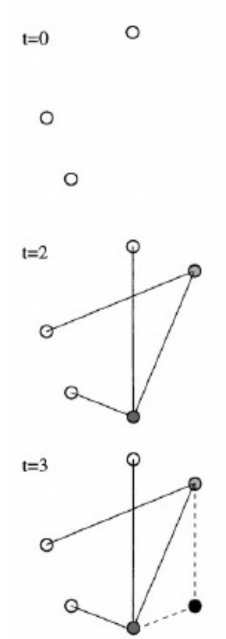
- Some types of graphs to consider:
  - Real graphs (social & otherwise)
  - Generated graphs:
    - **Erdos-Renyi “Bernoulli” or “Poisson”**
    - Watts-Strogatz “small world” graphs
    - Barbosi-Albert “preferential attachment” **generates scale-free graphs**
    - …

Barabasi-Albert Networks

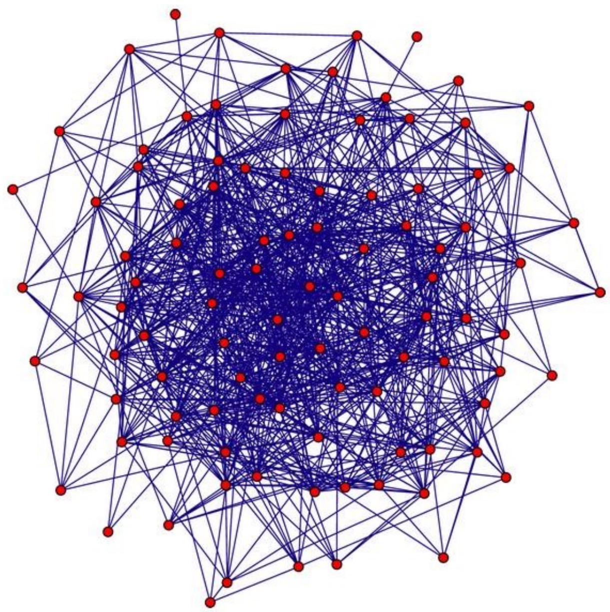
- *Science* **286** (1999)
- Start from a small number of node, add a new node with m links
- **Preferential Attachment**
  - Probability of these links to connect to existing nodes is proportional to the node's degree

$$\boxed{\prod(k_i) = \frac{k_i}{\sum_j k_j}}$$

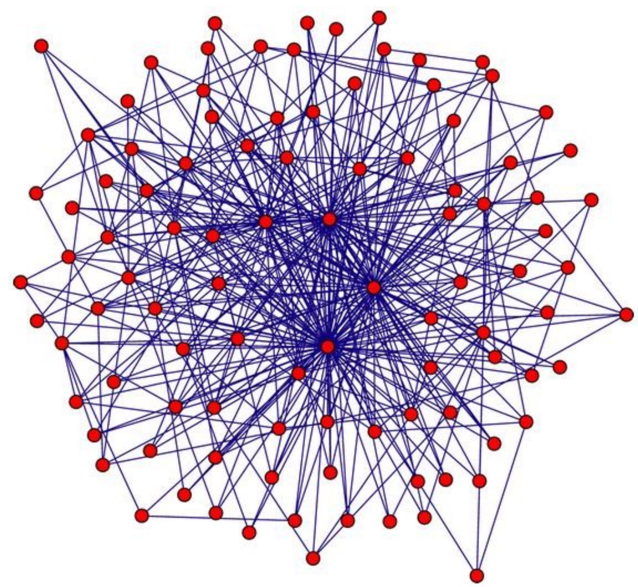
  - 'Rich gets richer'
- This creates 'hubs': few nodes with very large degrees



(a)



Random graph
(Erdos Renyi)



Preferential attachment
(Barabasi-Albert)

Graphs

- Some common properties of graphs:
  - **Distribution of node degrees: often scale-free**
  - Distribution of cliques (e.g., triangles)
  - **Distribution of paths**
    - **Diameter** (max shortest-path)
    - Effective **diameter** (90th percentile) **often small**
    - **Connected components usually one giant CC**
  - …

- Some types of graphs to consider:
  - Real graphs (social & otherwise)
  - Generated graphs:
    - **Erdos-Renyi "Bernoulli" or "Poisson"**
    - Watts-Strogatz "small world" graphs
    - Barbosi-Albert "preferential attachment" **generates scale-free graphs**
    - …

Homophily

- One definition: excess edges between similar nodes
- Another definition: excess edges between common neighbors of *v*

$$CC(v) = \frac{\#\,\text{triangles connected to } v}{\#\,\text{pairs connected to } v}$$

$$CC(V,E) = \frac{1}{|V|}\sum_{v} CC(v)$$

$$\boxed{CC'(V,E) = \frac{\#\,\text{triangles in graph}}{\#\,\text{length 3 paths in graph}}}$$

Homophily

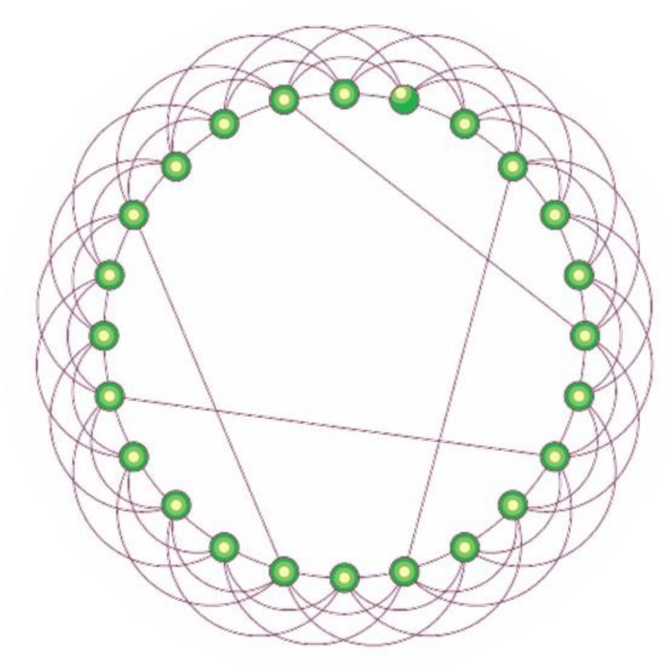
- In a random Erdos-Renyi graph:

$$CC'(V,E) = \frac{\#\,\text{triangles in graph}}{\#\,\text{length 3 paths in graph}} \approx \frac{1}{n}\ \text{for large n}$$

- Probably not realistic!
- In a natural graph, two of your mutual friends might also be friends
  - Both in the same class or organization
  - You introduced them
  - They introduced you

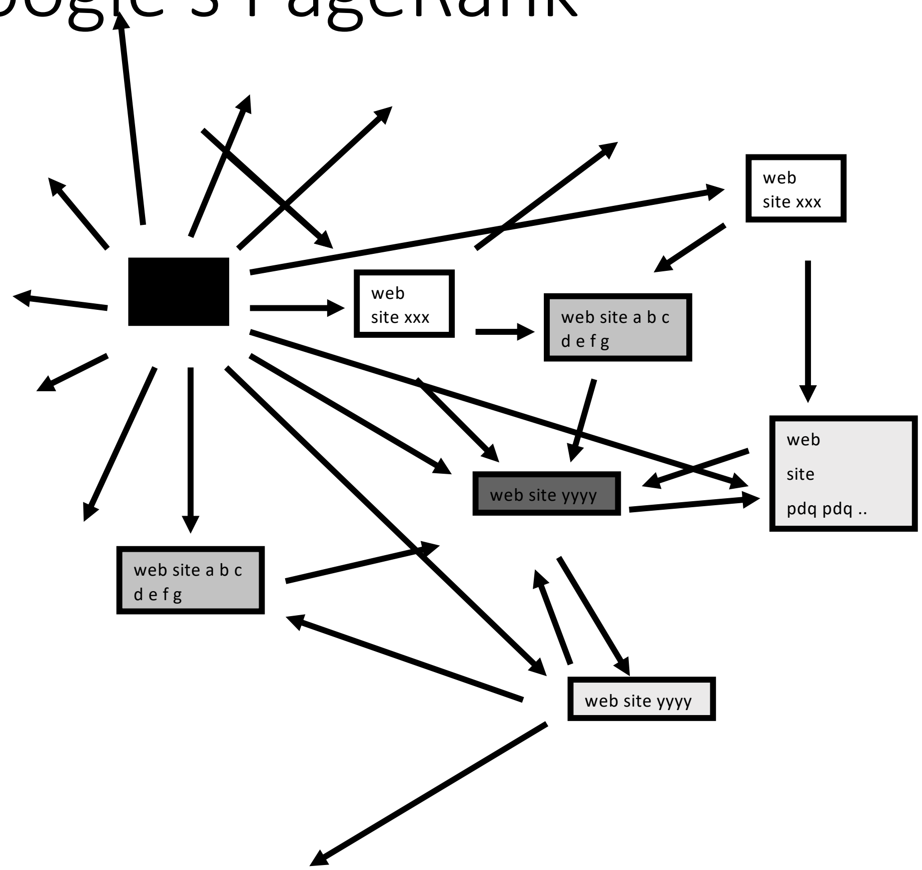
Watts-Strogatz model

- Start with a ring
- Connect each node to *k* nearest neighbors
    - ➔ homophily
- Add some random shortcuts from one point to another
    - ➔ small diameter
- Degree distribution *not* scale-free
- Generalizes to *d* dimensions



	network	type	n	m	z	ℓ	α	$C^{(1)}$	$C^{(2)}$
social	film actors	undirected	449 913	25 516 482	113.43	3.48	2.3	0.20	0.78
	company directors	undirected	7 673	55 392	14.44	4.60	–	0.59	0.88
	math coauthorship	undirected	253 339	496 489	3.92	7.57	–	0.15	0.34
	physics coauthorship	undirected	52 909	245 300	9.27	6.19	–	0.45	0.56
	biology coauthorship	undirected	1 520 251	11 803 064	15.53	4.92	–	0.088	0.60
	telephone call graph	undirected	47 000 000	80 000 000	3.16		2.1		
	email messages	directed	59 912	86 300	1.44	4.95	1.5/2.0		0.16
	email address books	directed	16 881	57 029	3.38	5.22	–	0.17	0.13
	student relationships	undirected	573	477	1.66	16.01	–	0.005	0.001
	sexual contacts	undirected	2 810				3.2		
information	WWW `nd.edu`	directed	269 504	1 497 135	5.55	11.27	2.1/2.4	0.11	0.29
	WWW Altavista	directed	203 549 046	2 130 000 000	10.46	16.18	2.1/2.7		
	citation network	directed	783 339	6 716 198	8.57		3.0/–		
	Roget's Thesaurus	directed	1 022	5 103	4.99	4.87	–	0.13	0.15
	word co-occurrence	undirected	460 902	17 000 000	70.13		2.7		0.44
technological	Internet	undirected	10 697	31 992	5.98	3.31	2.5	0.035	0.39
	power grid	undirected	4 941	6 594	2.67	18.99	–	0.10	0.080
	train routes	undirected	587	19 603	66.79	2.16	–		0.69
	software packages	directed	1 439	1 723	1.20	2.42	1.6/1.4	0.070	0.082
	software classes	directed	1 377	2 213	1.61	1.51	–	0.033	0.012
	electronic circuits	undirected	24 097	53 248	4.34	11.05	3.0	0.010	0.030
	peer-to-peer network	undirected	880	1 296	1.47	4.28	2.1	0.012	0.011
biological	metabolic network	undirected	765	3 686	9.64	2.56	2.2	0.090	0.67
	protein interactions	undirected	2 115	2 240	2.12	6.80	2.4	0.072	0.071
	marine food web	directed	135	598	4.43	2.05	–	0.16	0.23
	freshwater food web	directed	92	997	10.84	1.90	–	0.20	0.087
	neural network	directed	307	2 359	7.68	3.97	–	0.18	0.28

Google's PageRank



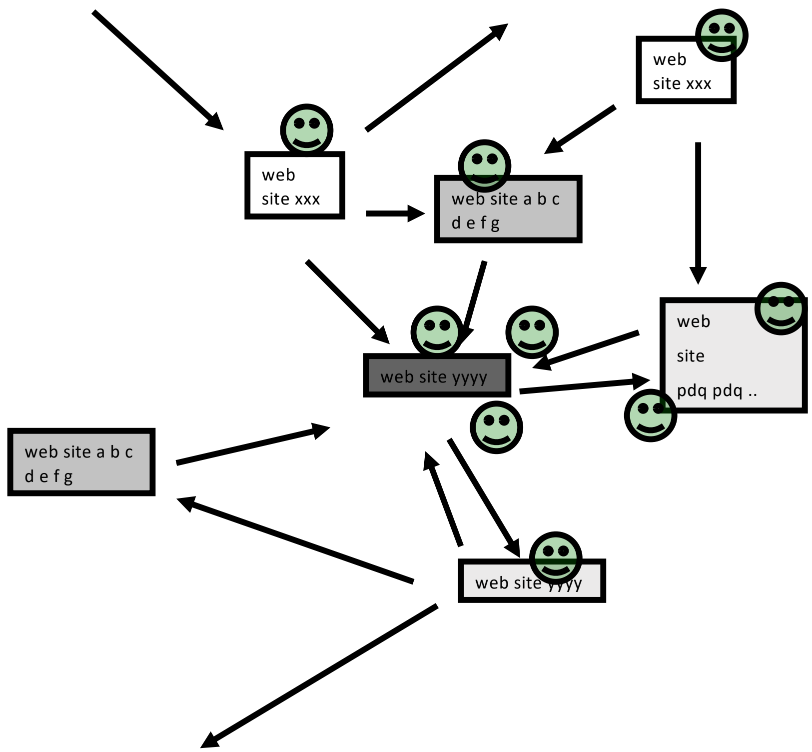
Inlinks are "good" (recommendations)

Inlinks from a "good" site are better than inlinks from a "bad" site

but inlinks from sites with many outlinks are not as "good"...

"Good" and "bad" are relative.

Google's PageRank

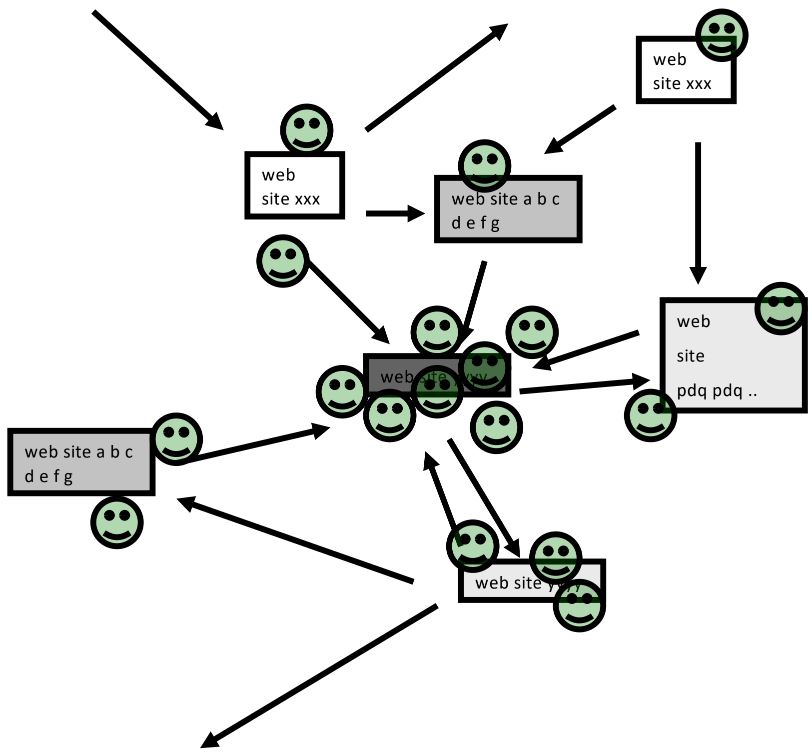


Imagine a "pagehopper" that always either 

- follows a random link, or
- jumps to random page

Google's PageRank

(Brin & Page, http://www-db.stanford.edu/~backrub/google.html)



Imagine a "pagehopper" that always either

- follows a random link, or
- jumps to random page

PageRank ranks pages by the amount of time the pagehopper spends on a page:

- or, if there were many pagehoppers, PageRank is the expected "crowd size"

Random Walks

G : a graph

P : transition probability matrix

$$P(u,v) = \begin{cases} \dfrac{1}{d_u} & if\ u :\ v, \\ 0 & otherwise. \end{cases}$$

d_u := the degree of u.

A lazy walk: $$W = \frac{I + P}{2}$$

Random Walks: PageRank

A (bored) surfer

- either surf a random webpage

  with probability α

- or surf a linked webpage

  with probability $1-\alpha$

α : the jumping constant

$$p = \alpha\left(\tfrac{1}{n}, \tfrac{1}{n}, \ldots., \tfrac{1}{n}\right) + (1-\alpha)\, pW$$

Random Walks: PageRank

Two equivalent ways to define PageRank $p = pr(\alpha, s)$

(1) $$p = \alpha s + (1-\alpha) pW$$

(2) $$p = \alpha \sum_{t=0}^{\infty} (1-\alpha)^t (sW^t)$$

$s = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ ⟹ the (original) PageRank

$s =$ some "seed", e.g., $(1, 0, \ldots, 0)$

⟹ personalized PageRank

Graph = Matrix

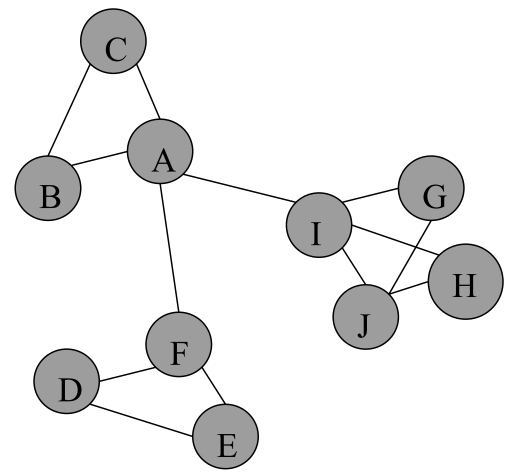
Vector = Node → Weight

M

	A	B	C	D	E	F	G	H	I	J
A	_	1	1			1				
B	1	_	1							
C	1	1	_							
D				_	1	1				
E				1	_	1				
F	1			1	1	_				
G							_		1	1
H								_	1	1
I							1	1	_	1
J							1	1	1	_

v

	A
A	3
B	2
C	3
D	
E	
F	
G	
H	
I	
J	



PageRank

- Let $\mathbf{u} = (1/N, \ldots, 1/N)$
  - dimension = #nodes N
- Let A = adjacency matrix: $[a_{ij}=1 \Leftrightarrow$ i links to j]
- Let W = $[w_{ij} = a_{ij}/\text{outdegree}(i)]$
  - w_{ij} is probability of jump from i to j
- Let $\mathbf{v}^0 = (1,1,\ldots.,1)$
  - or anything else you want
- Repeat until converged:
  - Let $\mathbf{v}^{t+1} = c\mathbf{u} + (1\text{-}c)\mathbf{W}\mathbf{v}^t$
    - c is probability of jumping “anywhere randomly”

Next: spectral clustering!