Linear Dynamical Systems

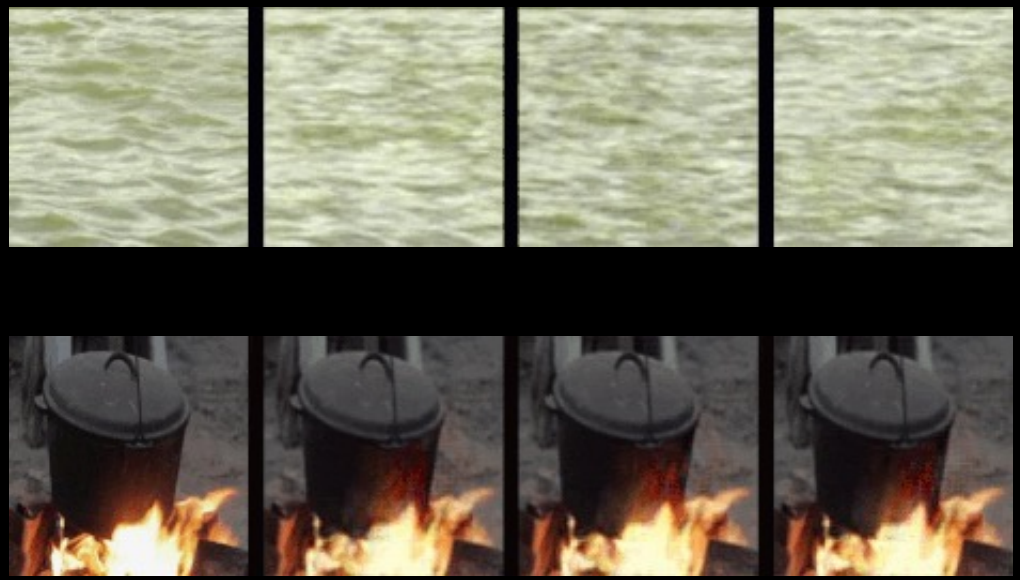
CSCI 4360/6360 Data Science II

Last time...

- Motion analysis via optical flow
- Parametric vs energy-based formulations
- Importance of assumptions
- Modern formulations
  - Robustness to outliers (large optical flow)
  - Relatedness to markov random fields
  - Coarse-to-fine image pyramids

Today

- A specific type of motion: **dynamic textures**



Dynamic textures

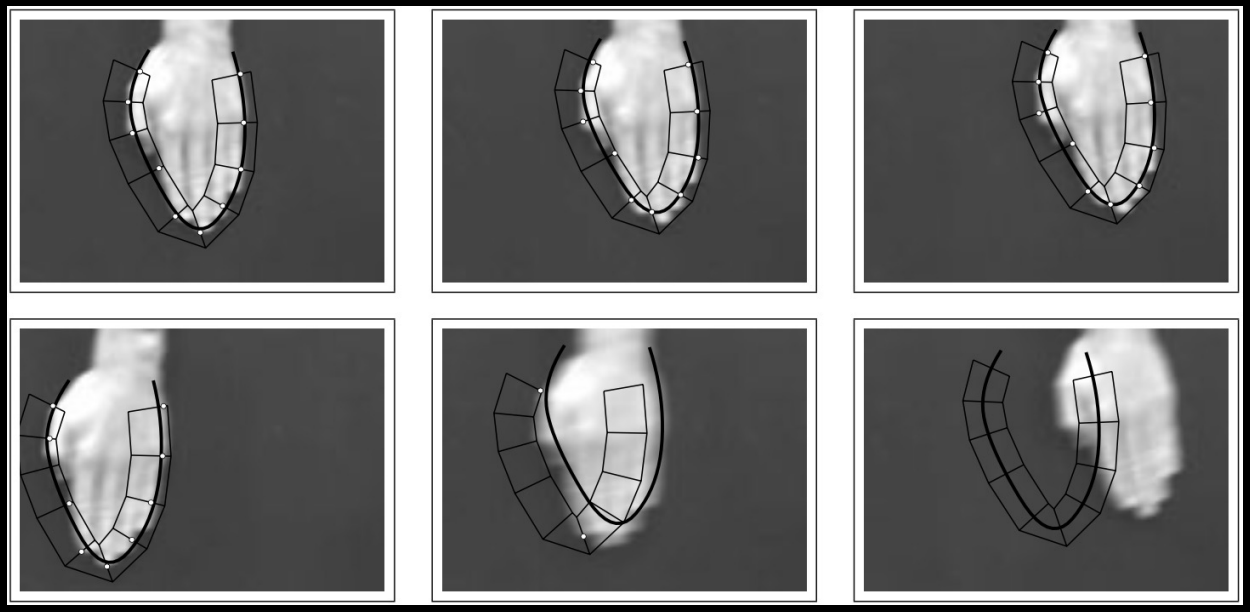
- *"Dynamic textured sequences are scenes with complex motion patterns due to interactions between multiple moving components."*
- Examples
  - Blowing leaves
  - Flickering flames
  - Water rippling
- **Multiple moving components: problematic for optical flow**
- How to analyze dynamic textures?

Dynamical Models

- Goal: an effective procedure for tracking changes over sequences of images, while maintaining a certain coherence of motions

Dynamical Models

- Hand tracking
- Top row: slow movements
- Bottom row: fast movements
- **Fixed curves or priors cannot exploit coherence of motion**



Linear Dynamical Models

- Two main components (using notation from Hyndman 2006):

Appearance Model

$$y_t = Cx_t + u_t$$

State Model

$$x_t = Ax_{t-1} + Wv_t$$

Autoregressive Models

- This is the definition of a 1st-order autoregressive (AR) process!

$$x_t = Ax_{t-1} + Wv_t$$

- Each observation (x_t) is a function of previous observations, plus some noise

- **Markov model!**

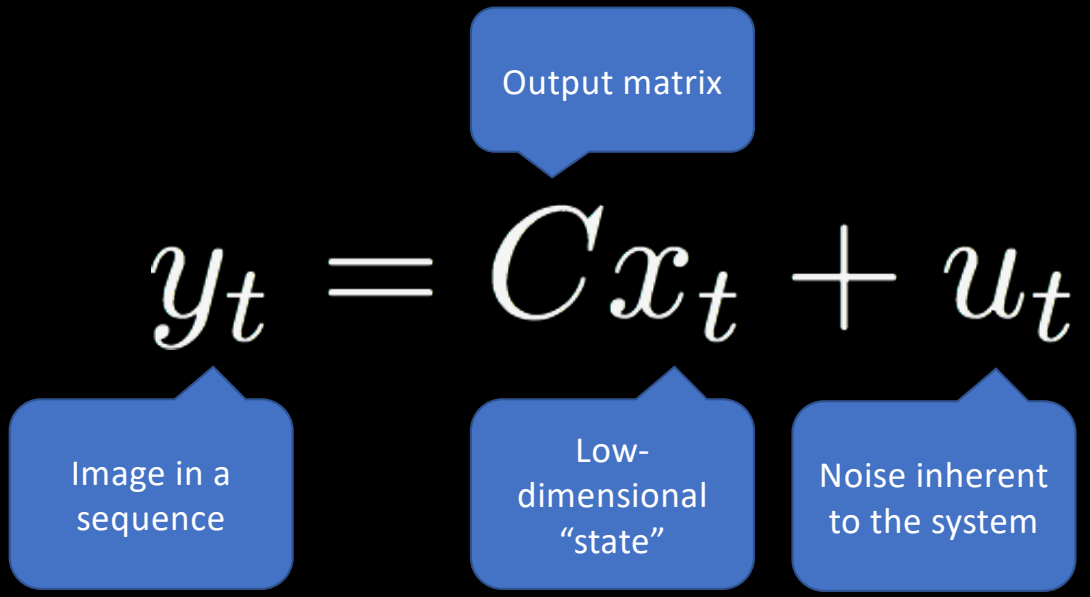
Autoregressive Models

- AR models can have higher orders than 1
- Each observation is dependent on the previous *d* observations

$$x_t = A_1 x_{t-1} + A_2 x_{t-2} + \ldots + A_d x_{t-d} + W v_t$$

Appearance Model

- y_t: image of height h and width w at time t, usually flattened into 1 x hw vector
- x_t: state space vector at time t, 1 x q (where q <<< hw)
- u_t: white Gaussian noise
- C: output matrix, maps between spaces, hw x q



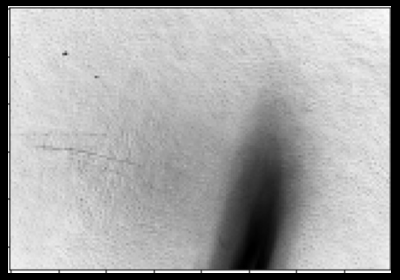
Appearance Model

$$y_t = Cx_t + u_t$$

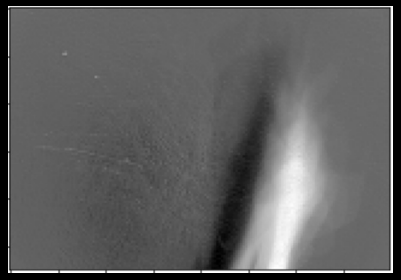
Each of these is 1 column of *C*.

There are *q* of them (first 4 shown here).

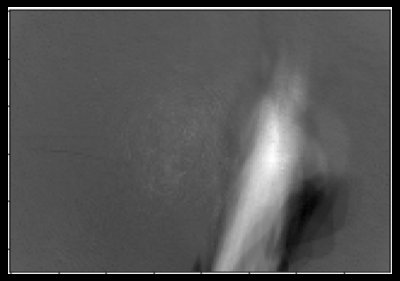
PC 1



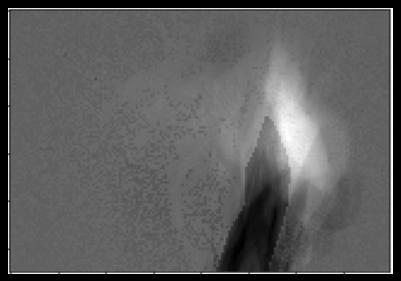
PC 2



PC 3



PC 4



Appearance Model

- How do we learn the appearance model?
- Choose state-space dimension size *q*
- Noise term is i.i.d Gaussian

$$Y = [\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_f]^T$$

$$Y = U\Sigma V^T$$

U-hat is a matrix of the first *q* columns of *U*

$$C = \hat{U}$$

$$X = \hat{\Sigma}\hat{V}^T$$

V-hat is a matrix of the first *q* columns of *V*, and sigma-hat is a diagonal matrix of the first *q* singular values

$$y_t = Cx_t + u_t$$

State Model

- x_t and x_{t-1}: state space vectors at times t and $t-1$, each 1 x q vector
- A: transition matrix, q x q matrix
- W: driving noise, q x q matrix
- v_t: white Gaussian noise

State transition matrix

$$x_t = Ax_{t-1} + Wv_t$$

Low-dimensional state at time t

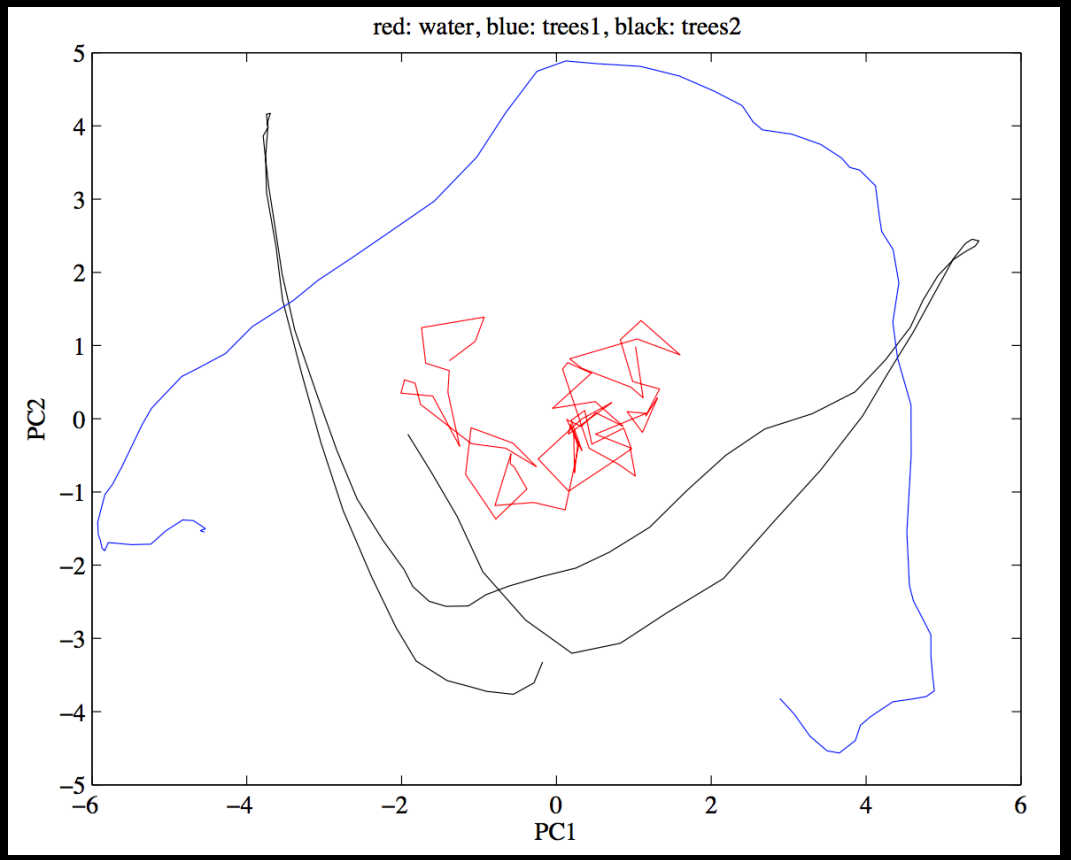
Low-dimensional state at $t - 1$

Driving noise

State Model

$$x_t = Ax_{t-1} + Wv_t$$

- Three textures
- *q = 2*



State Model

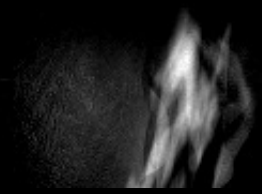
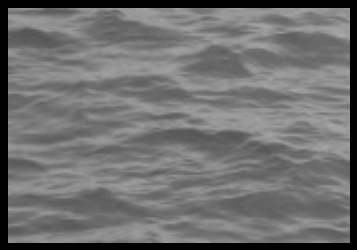
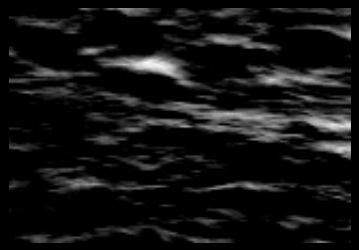
- How do we learn the state model?

$$x_t = Ax_{t-1} + Wv_t$$

- **Homework 2, ahoy!**

LDS as Generative Models

- Once we've learned the parameters, we can *generate new instances*

- Major strength of LDS!

Problems with LDS

- PCA = Linear + Gaussian
- What if the *state space* isn't linear, or data aren't Gaussian?

- Nonlinear appearance models
  - Wavelets
  - IsoMap
  - LLE
  - Kernel PCA
  - Laplacian Eigenmaps

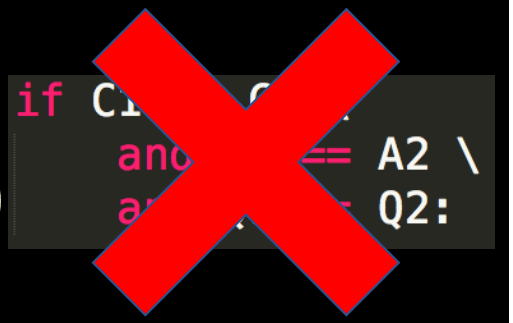
- These introduce their own problems!

Problems with LDS

- Comparing LDS models

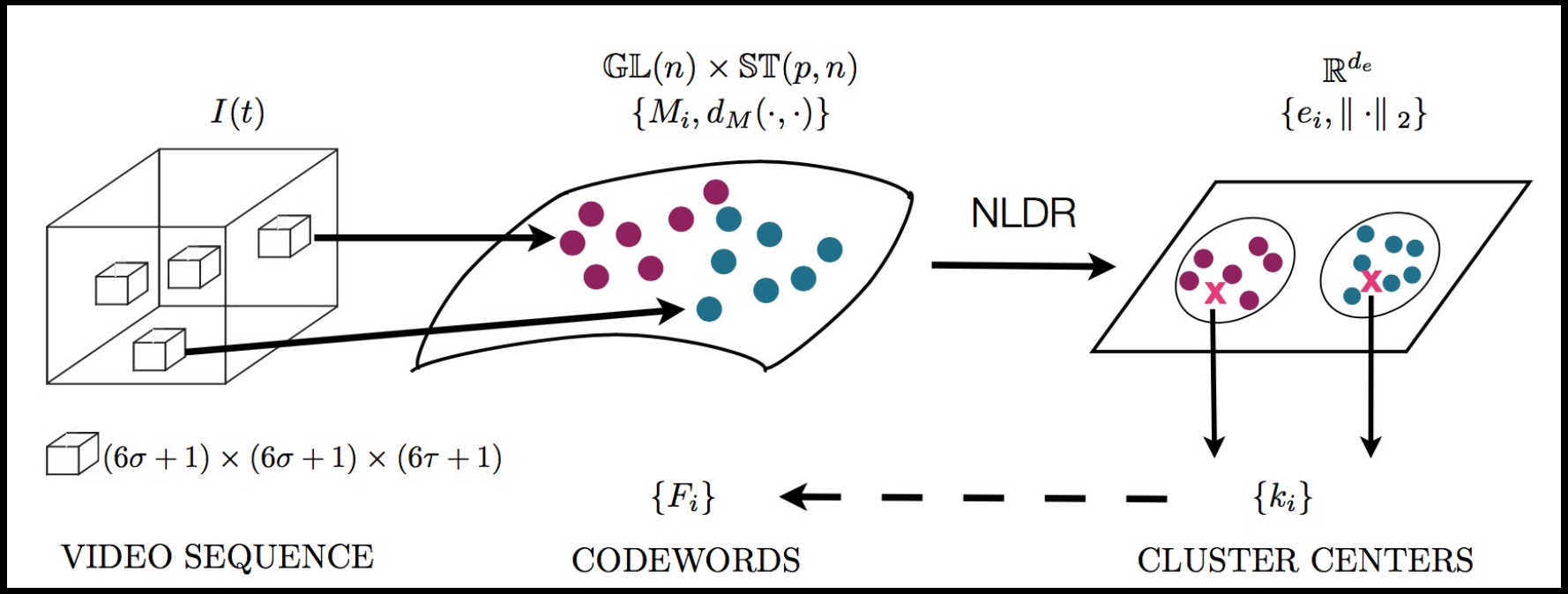
- Given a sequence *Y*:
- New sequence *Y'*:
- **How do we compare these systems?**
- Despite linear formulation, θ are NOT Euclidean
- Valid distance metrics include spectral methods and distribution comparators

$$\theta = (C, A, Q)$$
$$\theta' = (C', A', Q')$$



Comparing LDS

- Select multiple, non-overlapping patches from each video
- Build LDS for each patch



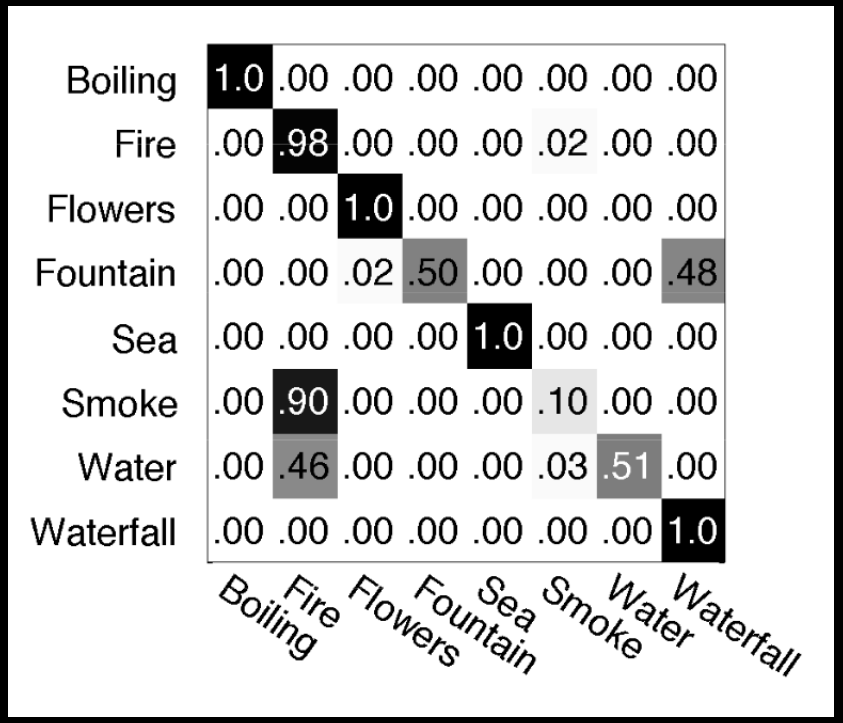
Comparing LDS

- Embed the LDS in low-dimensional space
  - We'll come back to this when we discuss embeddings!
- Compute cluster centroids in embedding space
  - These centroids become *codewords*
- Represent videos as a *document of codewords*
  - Compute TF-IDF
- Perform classification on document weight vectors

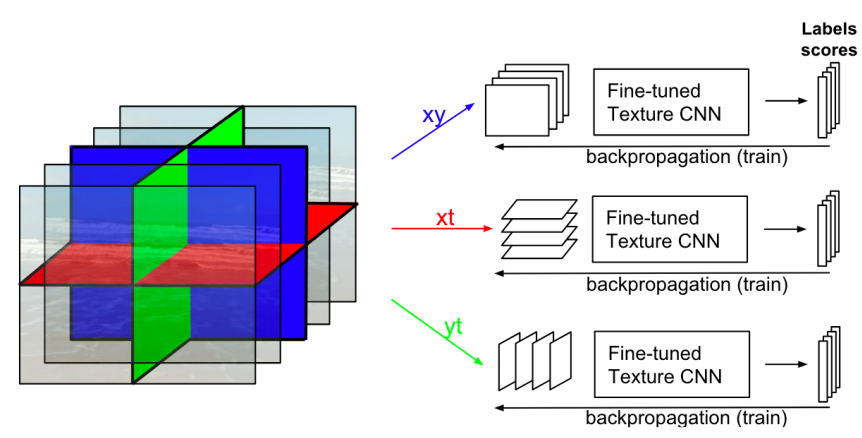
$$p = \arg\min_j \|e_j - k_i\|^2$$

$$w_{ik} = \frac{N_{ki}}{N_i} \ln\left(\frac{V}{V_i}\right)$$

Comparing LDS



Deep learning + dynamic textures



	boil	fire	flow.	pl.	fount	sea	sm.	wat.	wfall
boil	1	0	0	0	0	0	0	0	0
fire	0	0.97	0	0	0	0	0.03	0	0
flower	0	0	0.92	0	0.08	0	0	0	0
fount	0	0	0	0.96	0	0	0	0	0.04
plants	0	0	0	0	1	0	0	0	0
sea	0	0	0	0	0	0.97	0	0.03	0
smoke	0	0	0	0	0	0	0.75	0.25	0
water	0	0	0	0	0	0	0.03	0.97	0
wfalls	0	0	0	0.01	0	0	0	0	0.99

(a) xy **(b)** xt **(c)** yt

Figure 4.3: Examples of DT slices in three orthogonal planes of foliage, traffic and sea sequences from the DynTex database. (a) *xy* (spatial), (b) *xt* (temporal) and (c) *yt* (temporal).

Mamba state space models

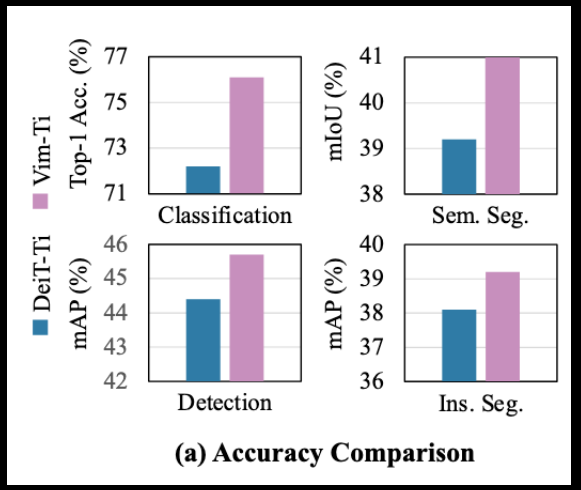
- Motivated by performance and space considerations of Transformer architectures in large language models
- (Conceptual) combination of RNN + CNN + AR models
- Innovations
  - Input selection mechanism
  - Hardware-aware algorithm
  - Architecture
- Vision Mamba (ViM) for image processing

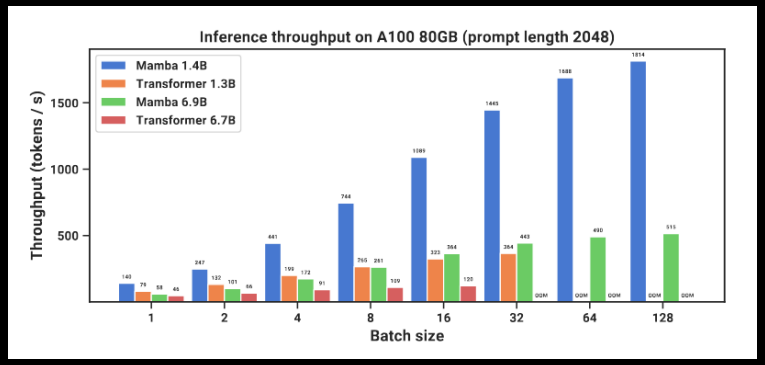
- We'll get into this more when we get to deep networks. For now:

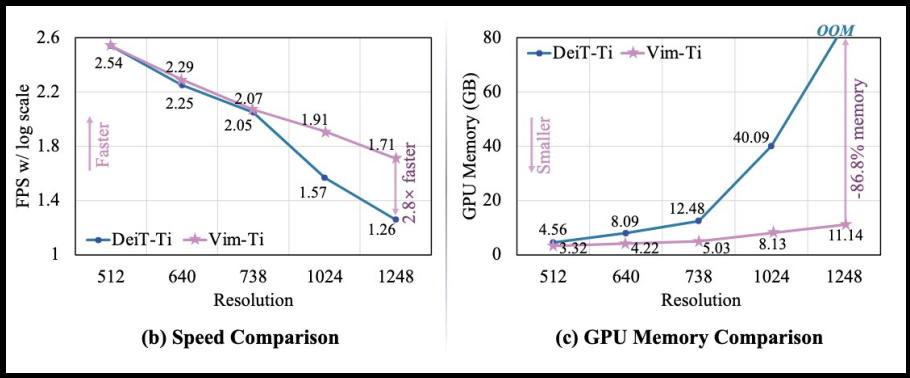
Mamba state space models

- Upshot: beat Transformer models of same parameter size, equivalent performance to Transformer models of 2x parameter size

- Much faster inference across the board (ie, generation) and with fewer resources







Conclusion

- Dynamic textures are motion with statistical regularity
- Regularity can be exploited through parametric representation
- Linear dynamical systems (LDS)
  - Autoregressive models (AR)
  - Markov assumption
  - Representation model + State model
  - Generative models
- Deep networks can learn the same feature set and in some cases exceed the performance of LDS (though are harder to train)
- Mamba state space models make LDS-like architectures cool again

References

- Hyndman *et al*, “Higher-order Autoregressive Models for Dynamic Textures”, BMVC 2007 http://www.cs.toronto.edu/~fleet/research/Papers/DynamicTextures.pdf
- Shumway and Stoffer, *Time Series Analysis and Its Applications* (3rd ed.), Chapters 3 and 6, http://www.db.ucsd.edu/static/TimeSeries.pdf
- Blake and Isard, *Active Contours*, Chapter 9-11, http://ww.vavlab.ee.boun.edu.tr/courses/574/material/books/blake_Active_Contours.pdf
- Doretto *et al*, “Dynamic Textures”, IJCV 2003 https://escholarship.org/uc/item/2m50f2fb
- Woolfe *et al*, “Shift Invariant Dynamic Texture Recognition”, ECCV 2006 https://link.springer.com/chapter/10.1007%2F11744047_42?LI=true
- Ravichandran *et al*, “View-invariant dynamic texture recognition using a bag of dynamical systems”, CVPR 2009 http://ieeexplore.ieee.org/abstract/document/5206847/
- Andrearczyk, Vincent. “Deep learning for texture and dynamic texture analysis”, 2020 https://doras.dcu.ie/22040/1/Vincent_Andrearczyk_Final_PhD_thesis.pdf
- Gu *et al*, “Mamba: Linear-Time Sequence Modeling with Selective State Spaces”, 2023 https://arxiv.org/abs/2312.00752
- Zhu *et al*, “Vision Mamba: Efficient Visual Representation Learning with Bidirectional State Space Model”, 2024 https://arxiv.org/abs/2401.09417