Randomized SVD

CSCI 4360/6360 Data Science II

Singular Value Decomposition (SVD)

- Given a (any!) matrix *M*, which is *n x m*, it can be represented as

$$M = U\Sigma V^T$$

- *U*: *n x n*, unitary matrix (orthogonal)
- Σ: *n x m*, diagonal matrix of singular values
- V^T: *m x m*, unitary matrix (orthogonal)

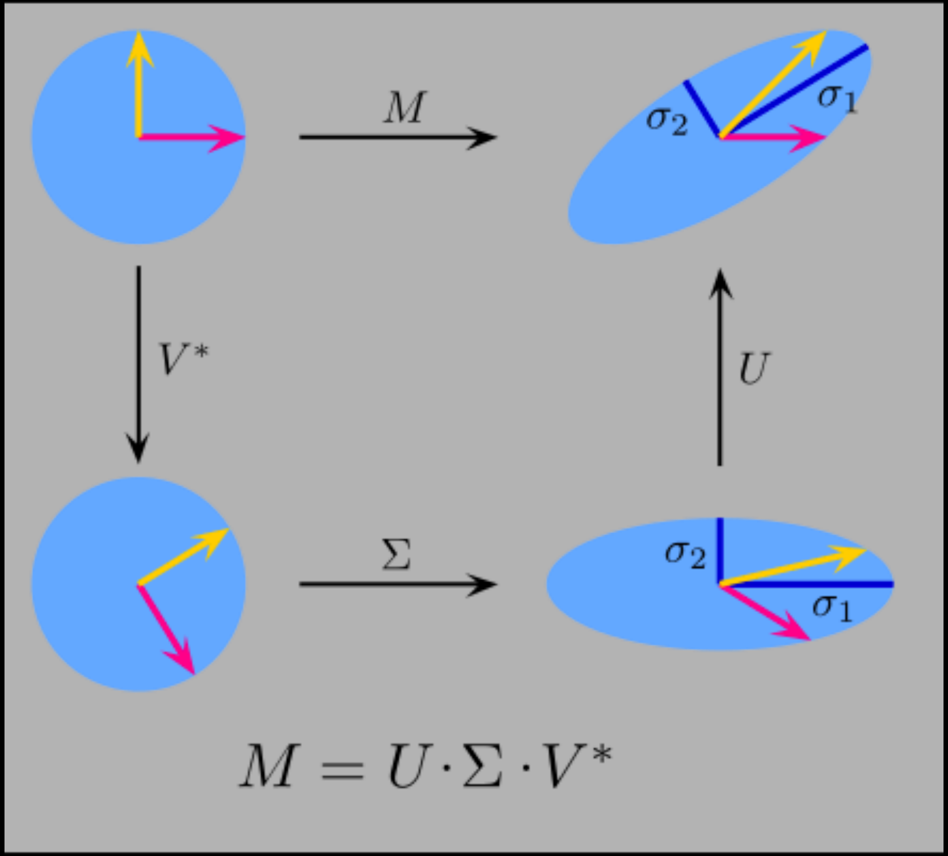
SVD

$$\boxed{\mathbf{A}_{[m \times n]} = \mathbf{U}_{[m \times r]}\, \Sigma_{[r \times r]}\, (\mathbf{V}_{[n \times r]})^T}$$

- **A**: **Input data matrix**
  - *m* x *n* matrix (e.g., *m* documents, *n* terms)
- **U**: **Left singular vectors**
  - *m* x *r* matrix (*m* documents, *r* concepts)
- **Σ**: **Singular values**
  - *r* x *r* diagonal matrix (strength of each ‘concept’)
    (*r* : rank of the matrix **A**)
- **V**: **Right singular vectors**
  - *n* x *r* matrix (*n* terms, *r* concepts)

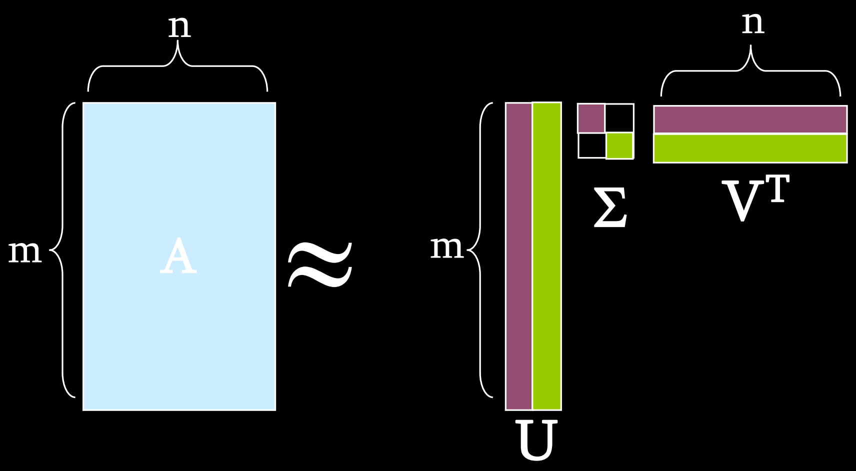
Singular Value Decomposition (SVD)

- Columns of *U* and *V* are orthonormal bases
- Singular values are the ”strength” of each singular vector



SVD

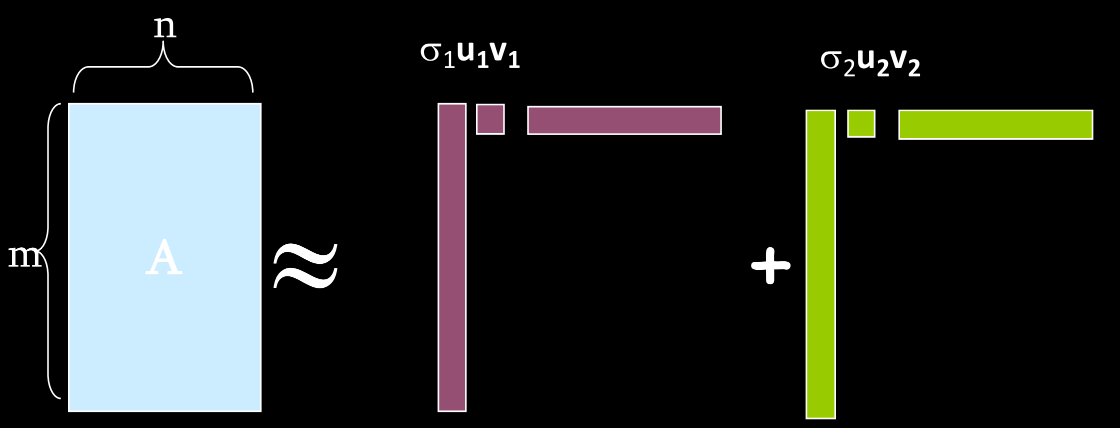
$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathrm{T}}$$



J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD

$$\mathbf{A} \approx \mathbf{U}\Sigma\mathbf{V}^T = \sum_i \sigma_i \mathbf{u}_i \circ \mathbf{v}_i^{\mathrm{T}}$$



σ_i … scalar
u_i … vector
v_i … vector

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD - Properties

It is **always** possible to decompose a real matrix $\boldsymbol{A}$ into $\boldsymbol{A = U \Sigma V^T}$, where

- $\boldsymbol{U, \Sigma, V}$: unique
- $\boldsymbol{U, V}$: column orthonormal
  - $\boldsymbol{U^T U = I; V^T V = I}$ ($\boldsymbol{I}$: identity matrix)
  - (Columns are orthogonal unit vectors)
- Σ: diagonal
  - Entries (**singular values**) are positive, and sorted in decreasing order ($\boldsymbol{\sigma_1 \geq \sigma_2 \geq \ldots \geq 0}$)

Nice proof of uniqueness: http://www.mpi-inf.mpg.de/~bast/ir-seminar-ws04/lecture2.pdf

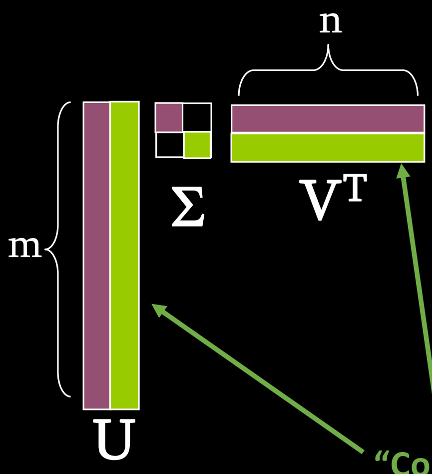
Why randomize SVD?

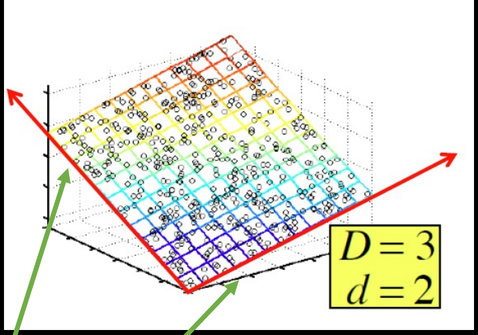
- **Runtime**
    - We’re good at generating [pseudo-]random numbers
    - Can easily parallelize / distribute matrix algebra
    - SVD, like PCA, runs $O(n^3)$, making anything beyond ~10^3 infeasible

SVD – Example: Users-to-Movies

- **A = U Σ V^T - example: Users to Movies**

$$\begin{array}{c|ccccc} & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\ \hline \text{SciFi} & 1 & 1 & 1 & 0 & 0 \\ & 3 & 3 & 3 & 0 & 0 \\ & 4 & 4 & 4 & 0 & 0 \\ & 5 & 5 & 5 & 0 & 0 \\ \text{Romance} & 0 & 2 & 0 & 4 & 4 \\ & 0 & 0 & 0 & 5 & 5 \\ & 0 & 1 & 0 & 2 & 2 \end{array} =$$





"Concepts"
AKA Latent dimensions
AKA Latent factors

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example: Users to Movies**

$$
\begin{array}{c|ccccc}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\hline
\uparrow & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
 & \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\text{SciFi} & \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\downarrow & \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
\uparrow & 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
\text{Romnce} & 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
\downarrow & 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD – Example: Users-to-Movies

- $A = U \Sigma V^T$ - example: **Users to Movies**

$$\begin{array}{cccccc} & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\ \text{SciFi} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ & \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ & \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ & \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ \text{Romnce} & 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ & 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ & 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{array} = \begin{array}{ccc} \text{SciFi-concept} & \text{Romance-concept} & \\ \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{array} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \mathbf{1.3} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:**

U **is "user-to-concept" similarity matrix**

SciFi-concept **Romance-concept**

$$
\begin{array}{c|ccccc}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\hline
\text{SciFi} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
 & \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
 & \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
 & \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
\text{Romnce} & 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
 & 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
 & 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:**

$$
\begin{array}{c|ccccc|}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\text{SciFi} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
 & \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
 & \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
 & \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
\text{Romnce} & 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
 & 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
 & 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{array}
=
\begin{array}{|ccc|}
\overset{\text{SciFi-concept}}{\downarrow} & & \\
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{array}
\times
\left[\begin{array}{ccc}
\overset{\text{“strength” of the SciFi-concept}}{\boxed{\mathbf{12.4}}} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{array}\right]
\times
$$

$$
\left[\begin{array}{ccccc}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{array}\right]
$$

SVD – Example: Users-to-Movies

- **$A = U \Sigma V^T$ - example:**

***V* is “movie-to-concept” similarity matrix**

SciFi-concept

$$
\begin{array}{c|ccccc}
 & \text{Matrix} & \text{Alien} & \text{Serenity} & \text{Casablanca} & \text{Amelie} \\
\hline
\text{SciFi} & \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
 & \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
 & \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
 & \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
\text{Romnce} & 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
 & 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
 & 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{array}
=
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \mathbf{1.3}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SciFi-concept

SVD - Interpretation #1

‘**movies**’, ‘**users**’ and ‘**concepts**’:

- ***U***: user-to-concept similarity matrix
- ***V***: movie-to-concept similarity matrix
- **Σ**: its diagonal elements:
  ‘strength’ of each concept

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD - Interpretation #2

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.13} & 0.02 & -0.01 \\ \mathbf{0.41} & 0.07 & -0.03 \\ \mathbf{0.55} & 0.09 & -0.04 \\ \mathbf{0.68} & 0.11 & -0.05 \\ 0.15 & \mathbf{-0.59} & \mathbf{0.65} \\ 0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\ 0.07 & \mathbf{-0.29} & \mathbf{0.32} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 & 0 \\ 0 & \mathbf{9.5} & 0 \\ 0 & 0 & \xcancel{\mathbf{1.3}} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\ 0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

SVD - Interpretation #2

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 & -0.01 \\
\mathbf{0.41} & 0.07 & -0.03 \\
\mathbf{0.55} & 0.09 & -0.04 \\
\mathbf{0.68} & 0.11 & -0.05 \\
0.15 & \mathbf{-0.59} & \mathbf{0.65} \\
0.07 & \mathbf{-0.73} & \mathbf{-0.67} \\
0.07 & \mathbf{-0.29} & \mathbf{0.32}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \cancel{\mathbf{1.3}}
\end{bmatrix}
\times
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
0.40 & \mathbf{-0.80} & 0.40 & 0.09 & 0.09
\end{bmatrix}
$$

SVD - Interpretation #2

More details

- **Q: How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$
\begin{bmatrix}
\mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\
\mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\
\mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\
\mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\
0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\
0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\
0 & 1 & 0 & \mathbf{2} & \mathbf{2}
\end{bmatrix}
\approx
\begin{bmatrix}
\mathbf{0.13} & 0.02 & \cancel{-0.01} \\
\mathbf{0.41} & 0.07 & \cancel{-0.03} \\
\mathbf{0.55} & 0.09 & \cancel{-0.04} \\
\mathbf{0.68} & 0.11 & \cancel{-0.05} \\
0.15 & \mathbf{-0.59} & \cancel{\mathbf{0.65}} \\
0.07 & \mathbf{-0.73} & \cancel{\mathbf{-0.67}} \\
0.07 & \mathbf{-0.29} & \cancel{\mathbf{0.32}}
\end{bmatrix}
\text{x}
\begin{bmatrix}
\mathbf{12.4} & 0 & 0 \\
0 & \mathbf{9.5} & 0 \\
0 & 0 & \cancel{\mathbf{1.3}}
\end{bmatrix}
\text{x}
\begin{bmatrix}
\mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\
0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \\
\cancel{0.40} & \cancel{\mathbf{-0.80}} & \cancel{0.40} & \cancel{0.09} & \cancel{0.09}
\end{bmatrix}
$$

SVD - Interpretation #2

More details

- Q: **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.13} & 0.02 \\ \mathbf{0.41} & 0.07 \\ \mathbf{0.55} & 0.09 \\ \mathbf{0.68} & 0.11 \\ 0.15 & \mathbf{-0.59} \\ 0.07 & \mathbf{-0.73} \\ 0.07 & \mathbf{-0.29} \end{bmatrix} \times \begin{bmatrix} \mathbf{12.4} & 0 \\ 0 & \mathbf{9.5} \end{bmatrix} \times \begin{bmatrix} \mathbf{0.56} & \mathbf{0.59} & \mathbf{0.56} & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & \mathbf{-0.69} & \mathbf{-0.69} \end{bmatrix}$$

SVD - Interpretation #2

More details

- Q: **How exactly is dim. reduction done?**
- **A: Set smallest singular values to zero**

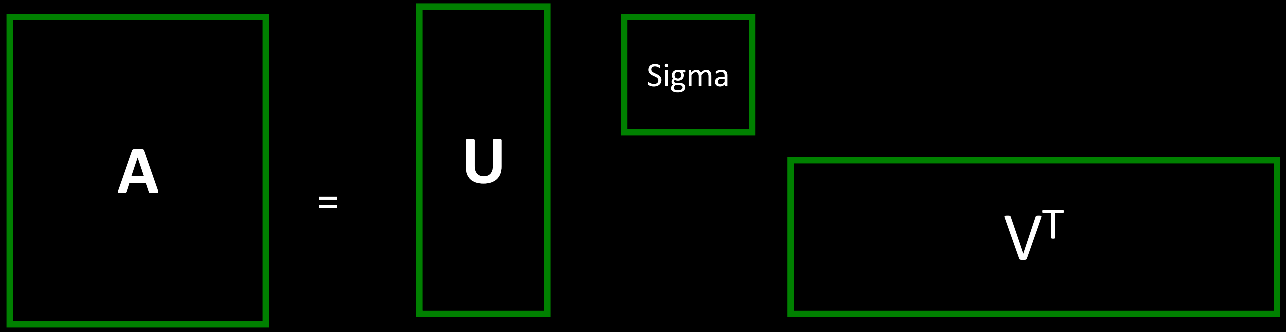
$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{3} & \mathbf{3} & \mathbf{3} & 0 & 0 \\ \mathbf{4} & \mathbf{4} & \mathbf{4} & 0 & 0 \\ \mathbf{5} & \mathbf{5} & \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 & \mathbf{4} & \mathbf{4} \\ 0 & 0 & 0 & \mathbf{5} & \mathbf{5} \\ 0 & \mathbf{1} & 0 & \mathbf{2} & \mathbf{2} \end{bmatrix} \approx \begin{bmatrix} \mathbf{0.92} & \mathbf{0.95} & \mathbf{0.92} & 0.01 & 0.01 \\ \mathbf{2.91} & \mathbf{3.01} & \mathbf{2.91} & -0.01 & -0.01 \\ \mathbf{3.90} & \mathbf{4.04} & \mathbf{3.90} & 0.01 & 0.01 \\ \mathbf{4.82} & \mathbf{5.00} & \mathbf{4.82} & 0.03 & 0.03 \\ 0.70 & \mathbf{0.53} & 0.70 & \mathbf{4.11} & \mathbf{4.11} \\ -0.69 & 1.34 & -0.69 & \mathbf{4.78} & \mathbf{4.78} \\ 0.32 & \mathbf{0.23} & 0.32 & \mathbf{2.01} & \mathbf{2.01} \end{bmatrix}$$

$\|A\text{-}B\|_F = \sqrt{\Sigma_{ij} (A_{ij}\text{-}B_{ij})^2}$
is “small”

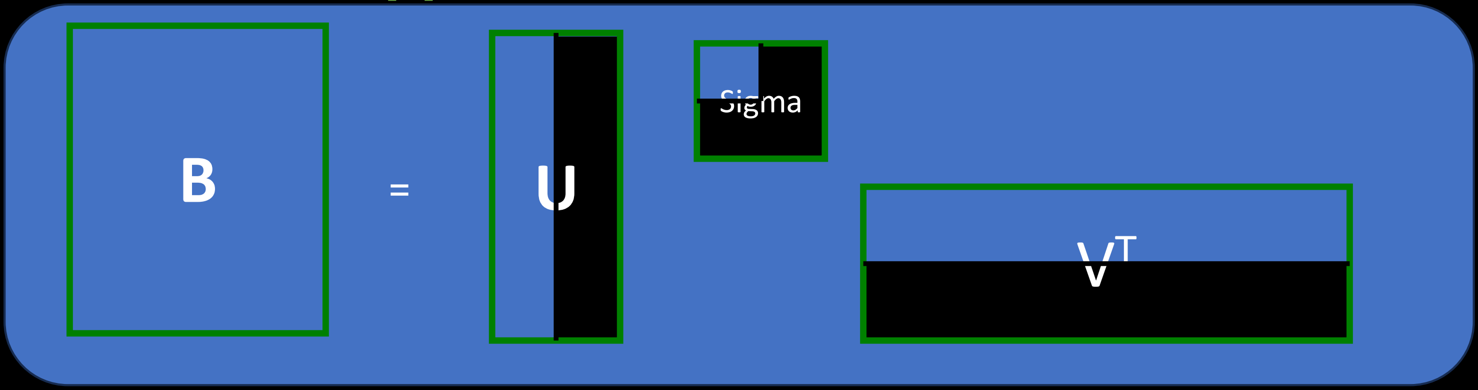
Frobenius norm:

$\|M\|_F = \sqrt{\Sigma_{ij}\, M_{ij}^2}$

SVD – Best Low Rank Approx.



B is best approximation of A



Relationship to PCA

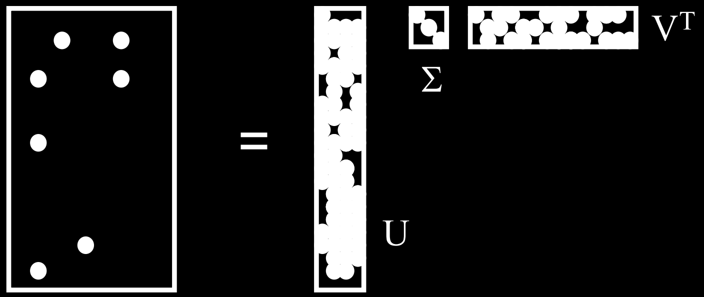
- SVD can be applied to *any* matrix; PCA only works on symmetric covariance matrices
- However, there is a relationship

$$M^T M = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T \Sigma)V^T$$
$$M M^T = U\Sigma V^T V \Sigma^T U^T = U(\Sigma \Sigma^T)U^T$$

- Columns of V are eigenvectors of $M^T M$
- Columns of U are eigenvectors of $M M^T$
- Singular values are square roots of eigenvalues of $M^T M$ or $M M^T$

SVD: Drawbacks

- + **Optimal low-rank approximation** in terms of Frobenius norm
- − **Interpretability problem:**
  - A singular vector specifies a linear combination of all input columns or rows
- − **Lack of sparsity:**
  - Singular vectors are **dense!**



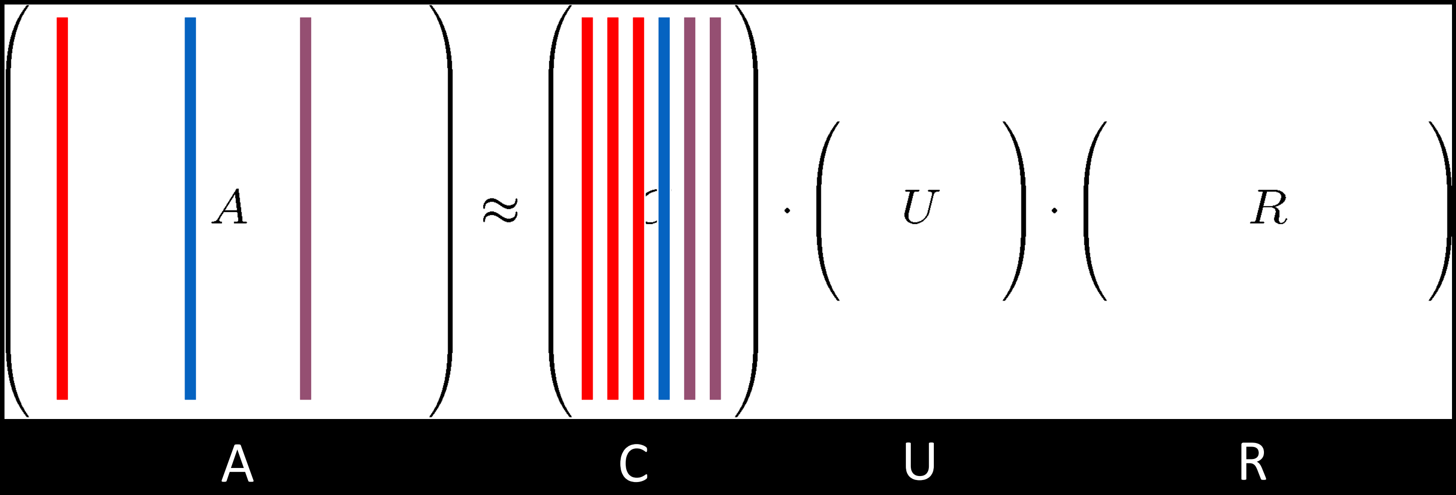
Frobenius norm:
$\|X\|_F = \sqrt{\Sigma_{ij} X_{ij}^2}$

CUR Decomposition

- **Goal: Express A as a product of matrices C,U,R**

  Make $\|A\text{-}C\cdot U\cdot R\|_F$ small

- **"Constraints" on C and R:**

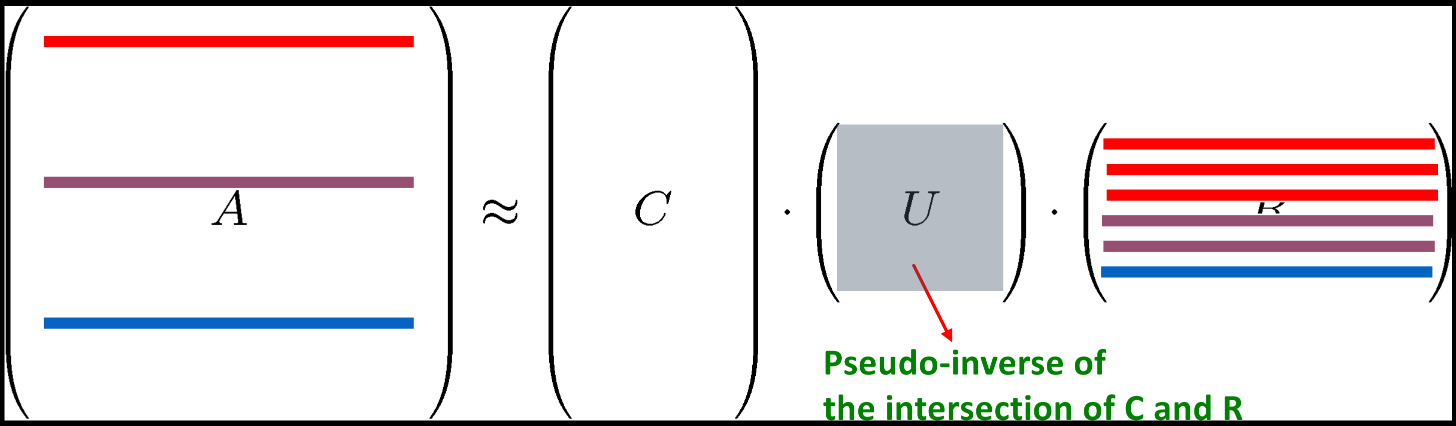


Frobenius norm:
$\|X\|_F = \sqrt{\Sigma_{ij} X_{ij}^2}$

CUR Decomposition

- **Goal: Express A as a product of matrices C,U,R**

  Make $\|A\text{-}C\cdot U\cdot R\|_F$ small
- **"Constraints" on C and R:**



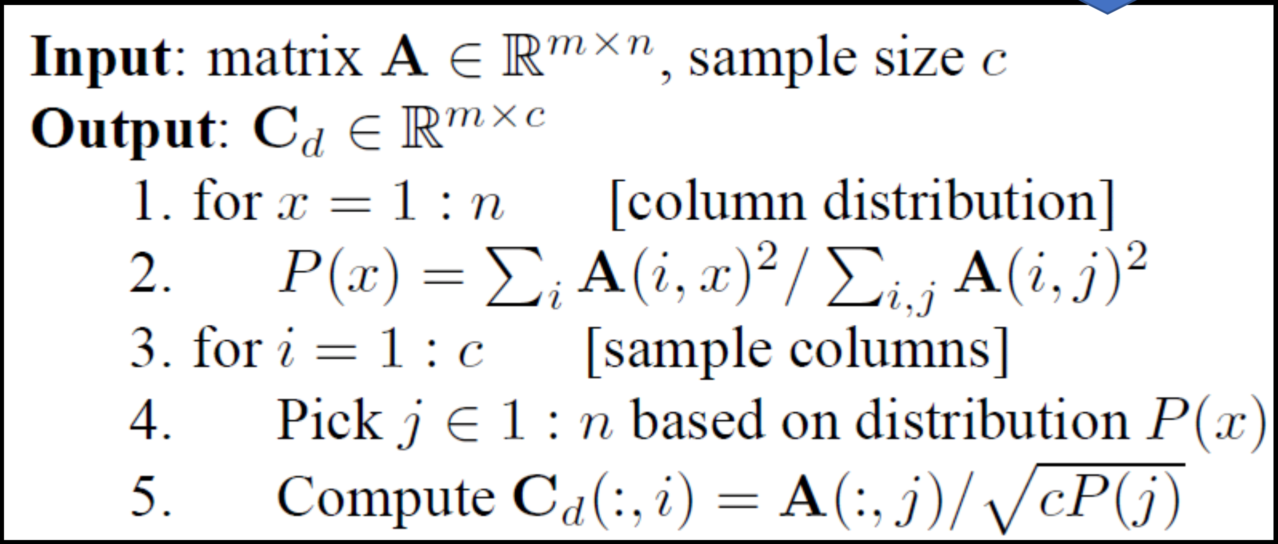
Pseudo-inverse of the intersection of C and R

A C U R

CUR: How it Works

Note this is a randomized algorithm; the same column can be sampled more than once

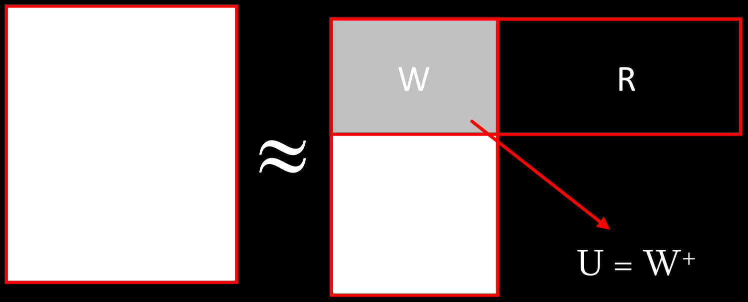
- **Sampling columns (similarly for rows):**



```
**Input**: matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$, sample size $c$
**Output**: $\mathbf{C}_d \in \mathbb{R}^{m \times c}$
    for $x = 1 : n$    [column distribution]
        $P(x) = \sum_i \mathbf{A}(i, x)^2 / \sum_{i,j} \mathbf{A}(i, j)^2$
    for $i = 1 : c$    [sample columns]
        Pick $j \in 1 : n$ based on distribution $P(x)$
        Compute $\mathbf{C}_d(:, i) = \mathbf{A}(:, j)/\sqrt{cP(j)}$
```

Computing U

- Let **W** be the "intersection" of sampled columns **C** and rows **R**
  - Let SVD of **W** = **X Z Y**^{T}
- **Then: U = W$^+$ = Y Z$^+$ X^T**
  - Z^+: **reciprocals of non-zero singular values:** $Z^+_{ii} = 1/ Z_{ii}$
  - W$^+$ is the "**pseudoinverse**"



Why pseudoinverse works?

W = X Z Y then $W^{-1} = X^{-1} Z^{-1} Y^{-1}$

Due to orthonomality

$X^{-1}=X^T$ and $Y^{-1}=Y^T$

Since Z is diagonal $Z^{-1} = 1/Z_{ii}$

Thus, if **W** is nonsingular, pseudoinverse is the true inverse

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

CUR: Pros & Cons

+ Easy interpretation

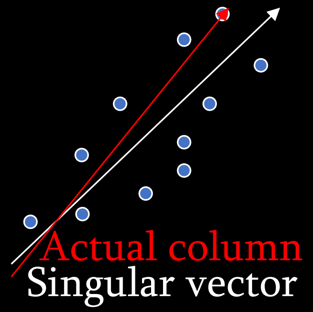
- Since the basis vectors are actual columns and rows

+ Sparse basis

- Since the basis vectors are actual columns and rows

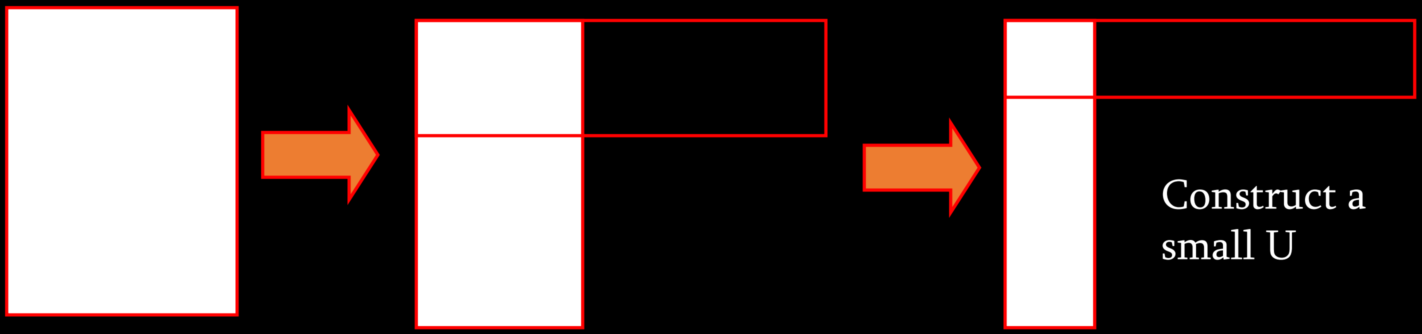
− Duplicate columns and rows

- Columns of large norms will be sampled many times



Solution

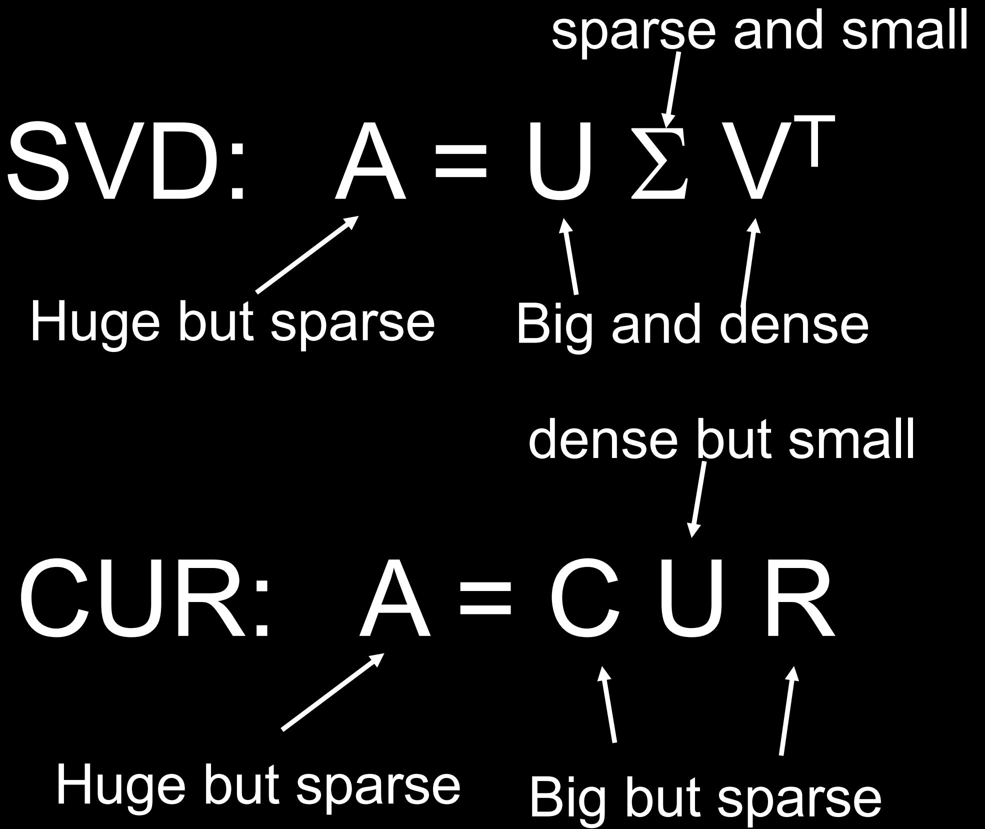
- **If we want to get rid of the duplicates:**
  - Throw them away
  - Scale (multiply) the columns/rows by the square root of the number of duplicates



Construct a small U

J. Leskovec, A. Rajaraman, J. Ullman: Mining of Massive Datasets, http://www.mmds.org

SVD vs. CUR



sparse and small

$$\text{SVD:}\quad A = U\ \Sigma\ V^T$$

Huge but sparse

Big and dense

dense but small

$$\text{CUR:}\quad A = C\ U\ R$$

Huge but sparse

Big but sparse

Stochastic SVD (SSVD)

- Uses **random projections** to find close approximation to SVD
- Combination of probabilistic strategies to maximize convergence likelihood
- Easily scalable to *massive* linear systems

Basic goal

- Matrix *A*
  - Find a low-rank approximation of *A*
  - Basic dimensionality reduction

$$\|A - QQ^{*}A\| < \epsilon$$

Preconditioning

Approximating range of *A*

- INPUT: *A*, *k*, *p*
- OUTPUT: *Q*

1. Draw Gaussian n x k test matrix Ω
2. Form product Y = AΩ
3. Orthogonalize columns of Y ➔Q

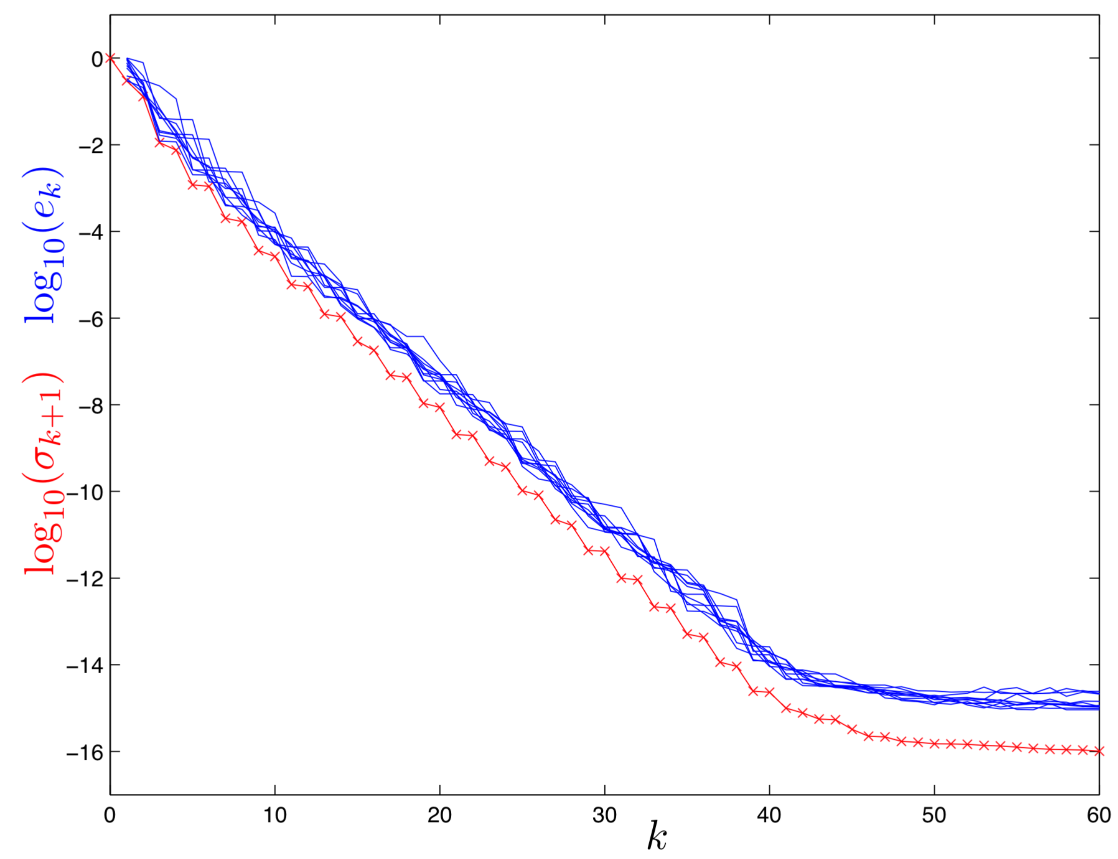
Approximating SVD of *A*

- INPUT: *Q*
- OUTPUT: Singular vectors *U*

1. Form k x n matrix $B = Q^T A$
2. Compute SVD of $B = \hat{U} \Sigma V^T$
3. Compute singular vectors $U = Q\hat{U}$

Empirical Results

- 1000x1000 matrix
- Several runs of empirical results (blue) to theoretical lower bound (red)
- **Error seems to be systemic**



Power iterations

- Affects decay of eigenvalues / singular values

$$Y = \xcancel{A}\Omega.$$

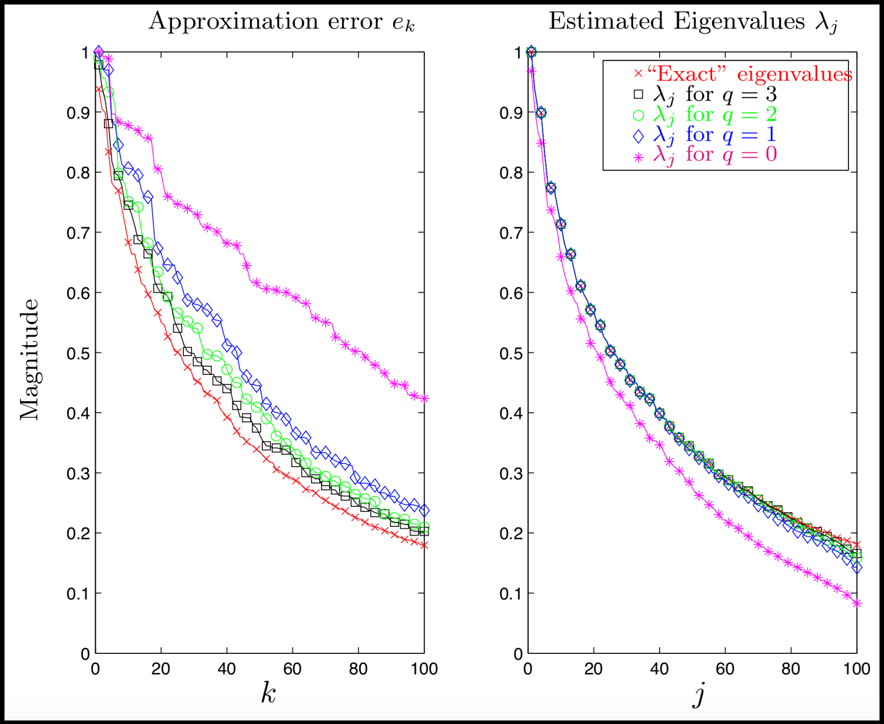
$$Y = (A A^*)^q A \Omega$$

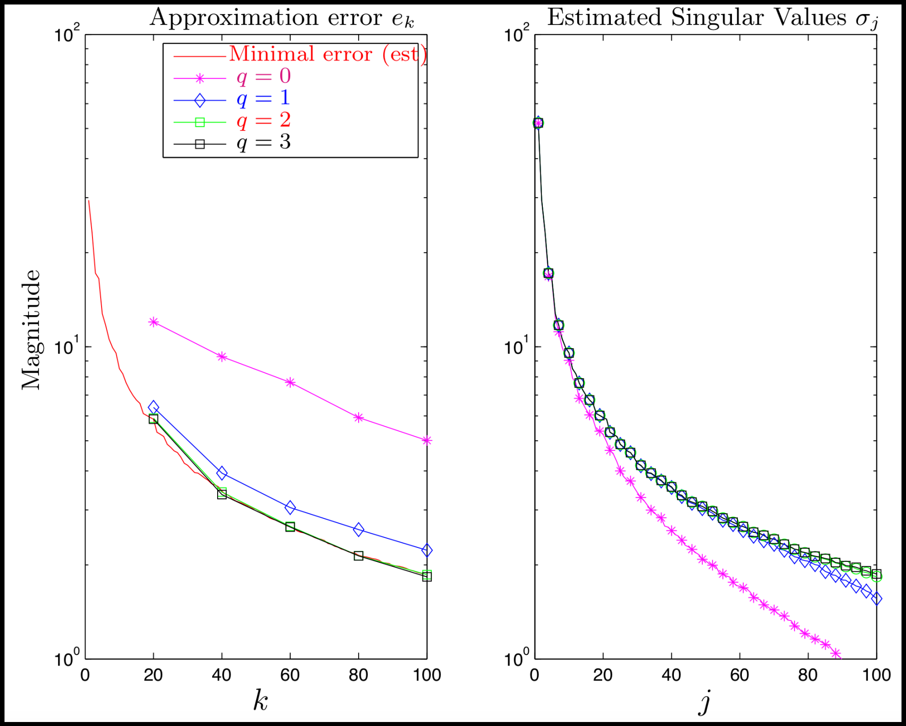
Power iterations

$$\begin{aligned}\mathbb{E}\|\mathsf{A}-\mathsf{Q}\mathsf{Q}^{\mathsf{T}}\mathsf{A}\|_2 &\leq \left(1+\sqrt{\frac{k}{p-1}}\right)\sigma_{k+1}+\frac{e\sqrt{k+p}}{p}\cdot\left(\sum_{j>k}\sigma_j^2\right)^{1/2}\\ &\leq \left[1+\frac{4\sqrt{k+p}}{p-1}\cdot\sqrt{\min\{m,n\}}\right]\sigma_{k+1}\\ &= C\cdot\sigma_{k+1}.\end{aligned}$$

Upshot: after only a single power iteration, the error is proportional to the *next* [uncomputed] singular value (times a constant *C*).

Empirical Results





Why does this work?

- Three primary reasons:

1. **Johnson-Lindenstrauss Lemma**
   - Low-dimensional embeddings preserve pairwise distances

$$(1-\varepsilon)\|u-v\|^2 \leq \|f(u)-f(v)\|^2 \leq (1+\varepsilon)\|u-v\|^2$$

2. **Concentration of measure**
   - Geometric interpretation of classical idea: regular functions of independent random variables rarely deviate far from their means
3. **Preconditioning**
   - Condition number: how much change in output is produced from change in input (relation to #1)
   - *Q* matrix lowers condition number while preserving overall system

$$\kappa = \frac{|\lambda_{\max}|}{|\lambda_{\min}|}$$

Summary

- Relationship of SVD and PCA
  - PCA: eigenvectors and eigenvalues of the covariance matrix (or kernel matrix, for Kernel PCA)
  - SVD: Low-rank approximation for *any* matrix
- CUR
  - Randomly sample columns of data matrix *A* to use as basis
  - Interpretable and sparse, but potentially oversample high-magnitude columns
- SVD via SGD
  - Reframe SVD as a matrix completion problem
  - Use SGD in alternating least-squares to infer ”missing” components
- SSVD
  - Full-blown Johnson-Lindenstrauss exploitation
  - Use random projections to approximate SVD to high accuracy
  - Requires some empirical tweaks (oversampling, power iterations)

References

- “Randomized methods for computing low-rank approximations of matrices”, https://amath.colorado.edu/faculty/martinss/Pubs/2012_halko_dissertation.pdf
- “CUR decomposition for compression and compressed sensing of large-scale traffic data”, https://dspace.mit.edu/openaccess-disseminate/1721.1/86879

[redacted]

Large-Scale Matrix Factorization with Distributed Stochastic Gradient Descent

Rainer Gemulla

talk pilfered from →

Peter J. Haas Yannis Sismanis Erik Nijkamp

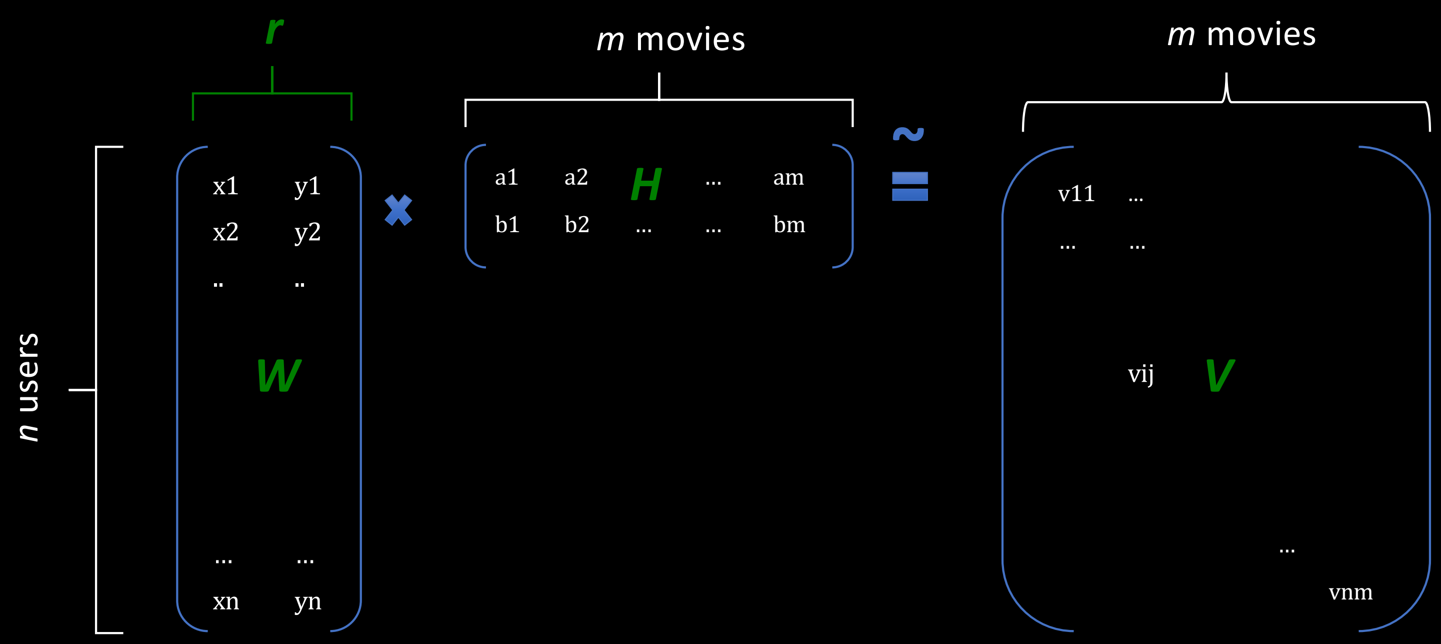
Collaborative Filtering

- Problem
  - Set of users
  - Set of items (movies, books, jokes, products, stories, ...)
  - Feedback (ratings, purchase, click-through, tags, ...)
- Predict additional items a user may like
  - Assumption: Similar feedback $\Longrightarrow$ Similar taste
- Example

$$\begin{array}{cc} & \begin{array}{ccc} \textit{Avatar} & \textit{The Matrix} & \textit{Up} \end{array} \\ \begin{array}{c} \textit{Alice} \\ \textit{Bob} \\ \textit{Charlie} \end{array} & \left(\begin{array}{ccc} ? & 4 & 2 \\ 3 & 2 & ? \\ 5 & ? & 3 \end{array} \right) \end{array}$$

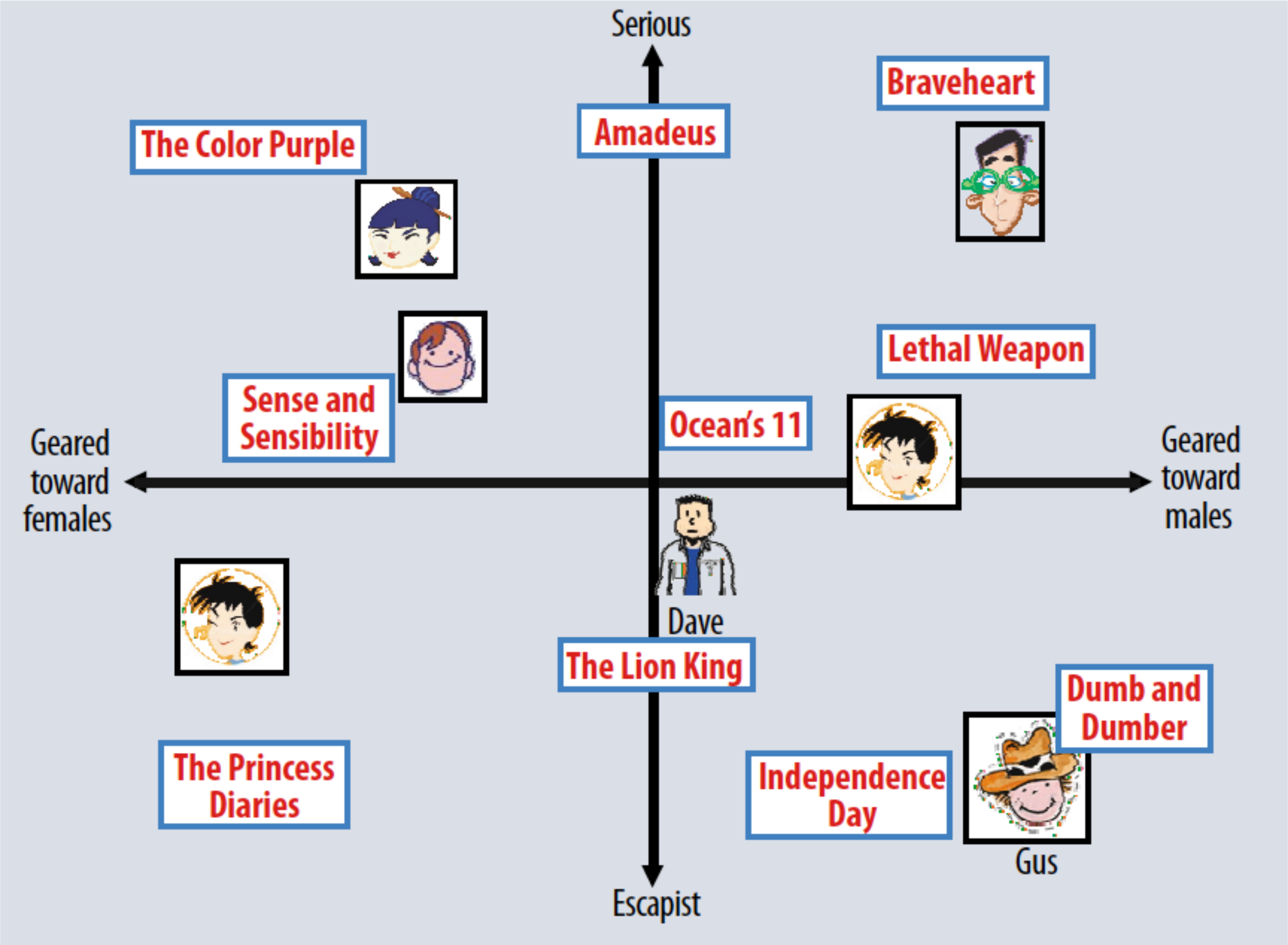
- Netflix competition: 500k users, 20k movies, 100M movie ratings, 3M question marks

Recovering latent factors in a matrix



V[i,j] = user i's rating of movie j

Semantic Factors (Koren et al., 2009)



Latent Factor Models

- Discover latent factors ($r = 1$)

	Avatar (*2.24*)	**The Matrix** (*1.92*)	**Up** (*1.18*)
Alice (*1.98*)		**4** (*3.8*)	**2** (*2.3*)
Bob (*1.21*)	**3** (*2.7*)	**2** (*2.3*)	
Charlie (*2.30*)	**5** (*5.2*)		**3** (*2.7*)

- Minimum loss

$$\min_{\mathbf{W},\mathbf{H}} \sum_{(i,j)\in Z} (\mathbf{V}_{ij} - [\mathbf{WH}]_{ij})^2$$

Latent Factor Models

- Discover latent factors ($r = 1$)

	Avatar (2.24)	**The Matrix** (1.92)	**Up** (1.18)
Alice (1.98)	? (4.4)	**4** (3.8)	**2** (2.3)
Bob (1.21)	**3** (2.7)	**2** (2.3)	? (1.4)
Charlie (2.30)	**5** (5.2)	? (4.4)	**3** (2.7)

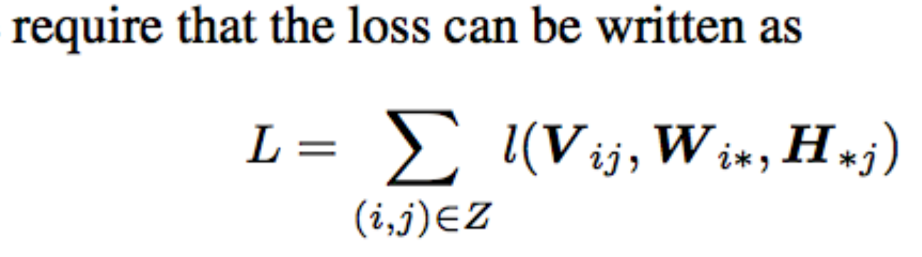
- Minimum loss

$$\min_{\mathbf{W},\mathbf{H},\mathbf{u},\mathbf{m}} \sum_{(i,j)\in Z} (\mathbf{V}_{ij} - \mu - \mathbf{u}_i - \mathbf{m}_j - [\mathbf{WH}]_{ij})^2$$

$$+ \lambda (\|\mathbf{W}\| + \|\mathbf{H}\| + \|\mathbf{u}\| + \|\mathbf{m}\|)$$

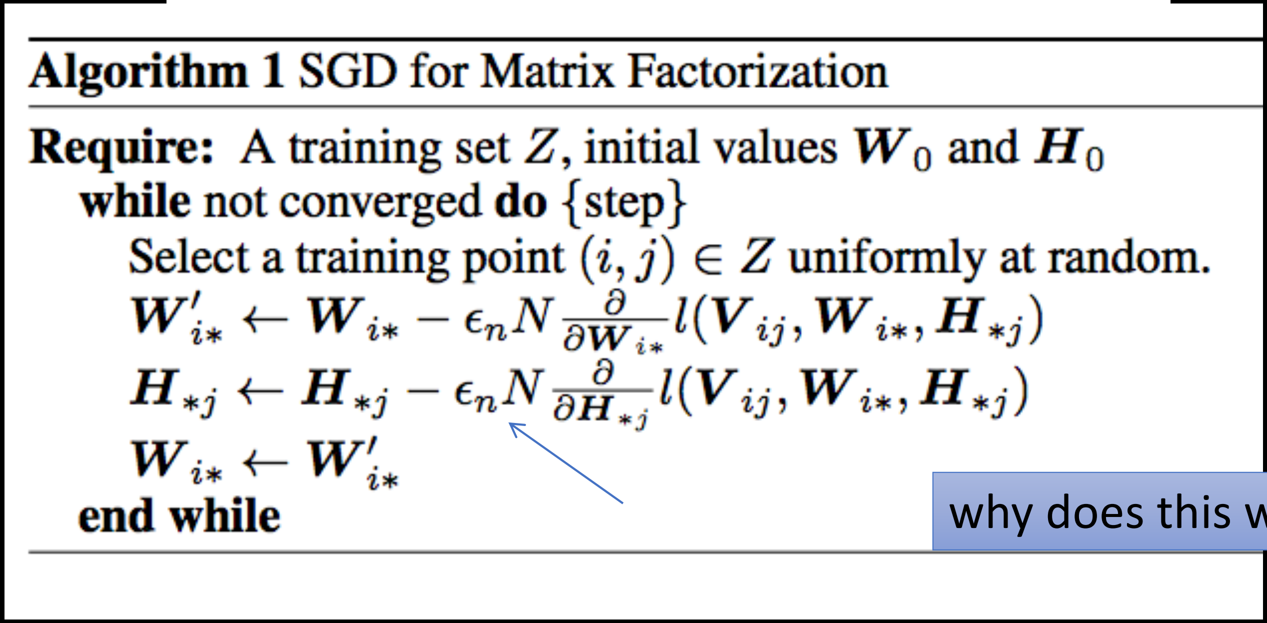
- Bias, regularization

Matrix factorization as SGD



require that the loss can be written as

$$L = \sum_{(i,j) \in Z} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j})$$



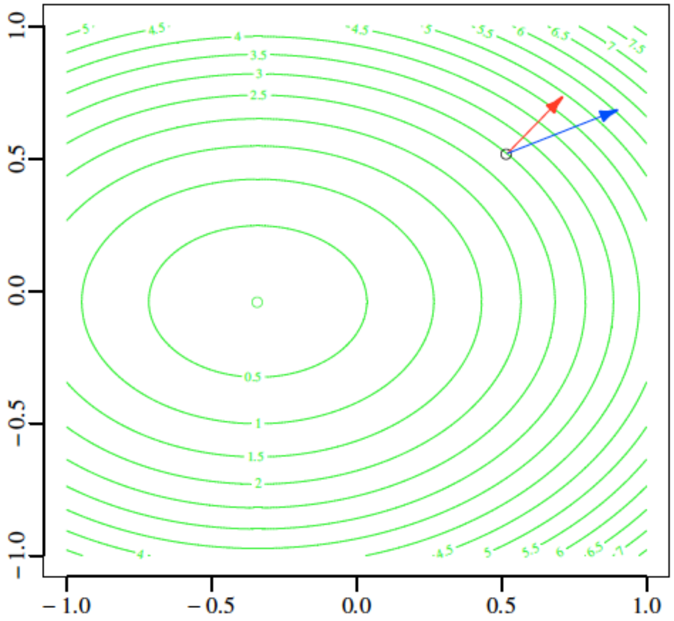
Algorithm 1 SGD for Matrix Factorization

```
Require: A training set $Z$, initial values $W_0$ and $H_0$
  while not converged do {step}
    Select a training point $(i, j) \in Z$ uniformly at random.
    $W'_{i*} \leftarrow W_{i*} - \epsilon_n N \frac{\partial}{\partial W_{i*}} l(V_{ij}, W_{i*}, H_{*j})$
    $H_{*j} \leftarrow H_{*j} - \epsilon_n N \frac{\partial}{\partial H_{*j}} l(V_{ij}, W_{i*}, H_{*j})$
    $W_{i*} \leftarrow W'_{i*}$
  end while
```

why does this work?

Stochastic Gradient Descent

- Find minimum θ^* of function L
- Pick a starting point θ_0
- Approximate gradient $\hat{L}'(\theta_0)$

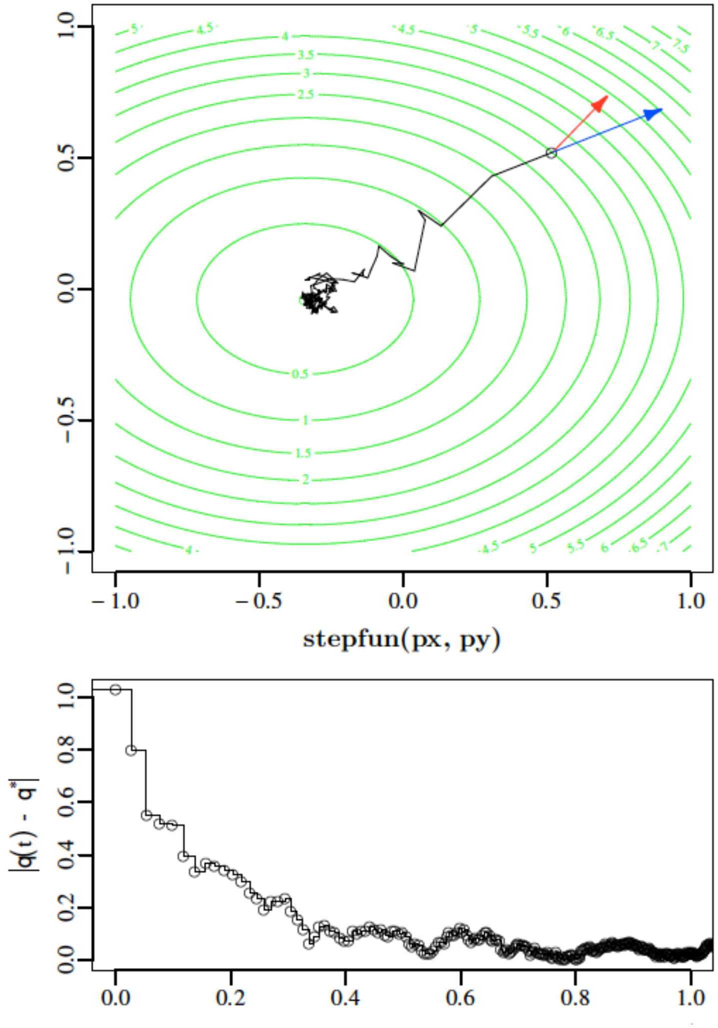


Stochastic Gradient Descent

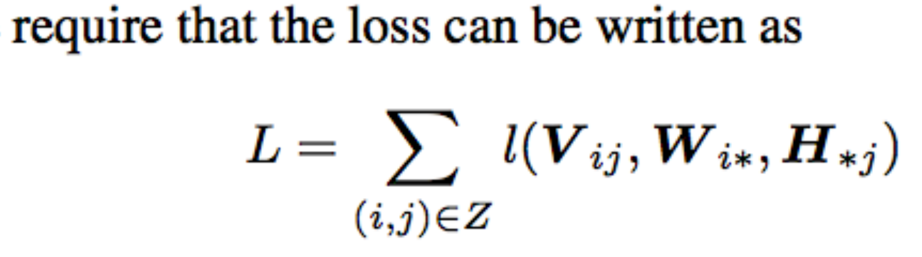
- Find minimum θ^* of function L
- Pick a starting point θ_0
- Approximate gradient $\hat{L}'(\theta_0)$
- Jump "approximately" downhill
- Stochastic difference equation

$$\theta_{n+1} = \theta_n - \epsilon_n \hat{L}'(\theta_n)$$

- Under certain conditions, asymptotically approximates (continuous) gradient descent

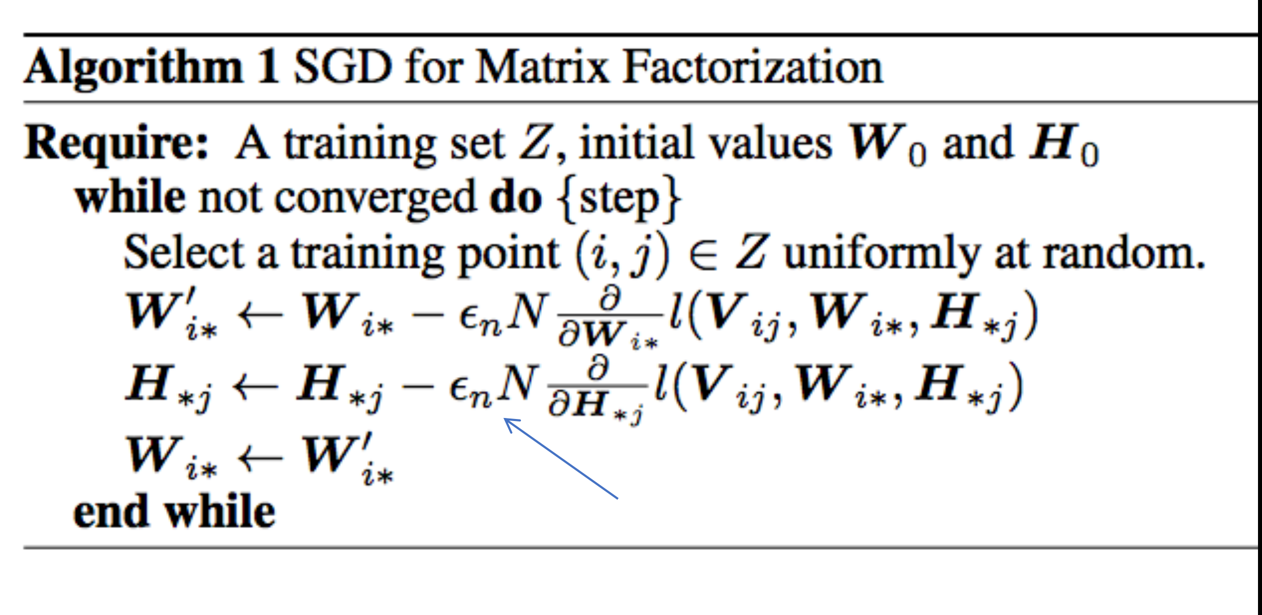


Why does this work?



require that the loss can be written as

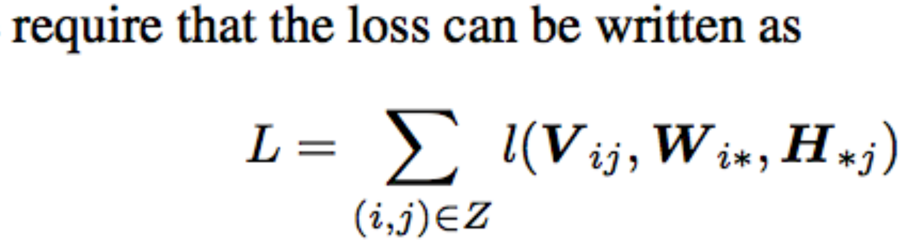
$$L = \sum_{(i,j)\in Z} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j})$$



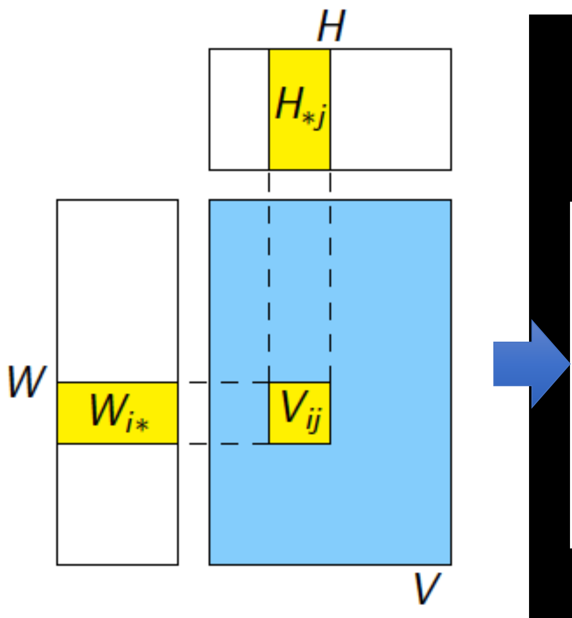
Algorithm 1 SGD for Matrix Factorization

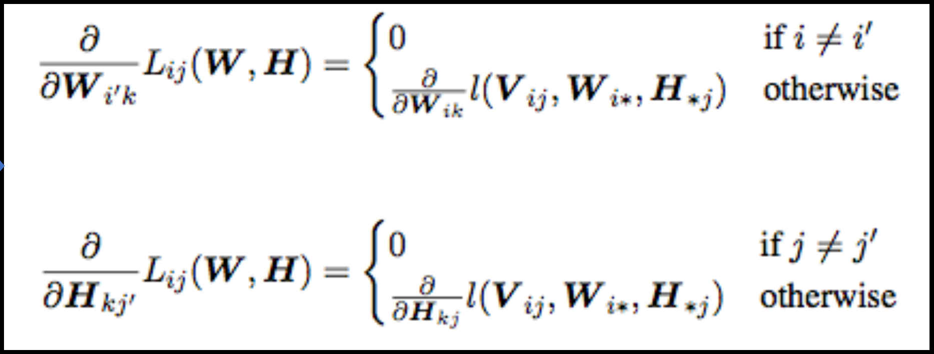
```
Require: A training set $Z$, initial values $\boldsymbol{W}_0$ and $\boldsymbol{H}_0$
  while not converged do {step}
    Select a training point $(i, j) \in Z$ uniformly at random.
    $\boldsymbol{W}'_{i*} \leftarrow \boldsymbol{W}_{i*} - \epsilon_n N \frac{\partial}{\partial \boldsymbol{W}_{i*}} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j})$
    $\boldsymbol{H}_{*j} \leftarrow \boldsymbol{H}_{*j} - \epsilon_n N \frac{\partial}{\partial \boldsymbol{H}_{*j}} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j})$
    $\boldsymbol{W}_{i*} \leftarrow \boldsymbol{W}'_{i*}$
  end while
```

Key Claim


require that the loss can be written as

$$L = \sum_{(i,j) \in Z} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j})$$




$$\frac{\partial}{\partial \boldsymbol{W}_{i'k}} L_{ij}(\boldsymbol{W}, \boldsymbol{H}) = \begin{cases} 0 & \text{if } i \neq i' \\ \frac{\partial}{\partial \boldsymbol{W}_{ik}} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j}) & \text{otherwise} \end{cases}$$

$$\frac{\partial}{\partial \boldsymbol{H}_{kj'}} L_{ij}(\boldsymbol{W}, \boldsymbol{H}) = \begin{cases} 0 & \text{if } j \neq j' \\ \frac{\partial}{\partial \boldsymbol{H}_{kj}} l(\boldsymbol{V}_{ij}, \boldsymbol{W}_{i*}, \boldsymbol{H}_{*j}) & \text{otherwise} \end{cases}$$

Checking the claim

$$\frac{\partial}{\partial \boldsymbol{W}_{i*}} L(\boldsymbol{W}, \boldsymbol{H}) = \frac{\partial}{\partial \boldsymbol{W}_{i*}} \sum_{(i',j) \in Z} L_{i'j}(\boldsymbol{W}_{i'*}, \boldsymbol{H}_{*j}) = \sum_{j \in Z_{i*}} \frac{\partial}{\partial \boldsymbol{W}_{i*}} L_{ij}(\boldsymbol{W}_{i*}, \boldsymbol{H}_{*j}),$$

where $Z_{i*} = \{\, j \colon (i, j) \in Z \,\}$.

$$\frac{\partial}{\partial \boldsymbol{H}_{*j}} L(\boldsymbol{W}, \boldsymbol{H}) = \sum_{i \in Z_{*j}} \frac{\partial}{\partial \boldsymbol{W}_{*j}} L_{ij}(\boldsymbol{W}_{i*}, \boldsymbol{H}_{*j}),$$

where $Z_{*j} = \{\, i \colon (i, j) \in Z \,\}$.

Think for SGD for logistic regression

- LR loss = compare *y* and *ŷ* = dot(w,x)
- similar but now update w (user weights) and x (movie weight)

Stochastic Gradient Descent on Netflix Data